Specified Recovery *

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Abstract

The Recovery Theorem (Ross, 2013) establishes a set of sufficient conditions for the unique resolution of the market's subjective belief and its risk preference. We show that the implementation of the Recovery Theorem via fitting an effective Arrow-Debreu (AD) price matrix and the market's characteristics recovered thereof depend endogenously and crucially on a subjectively specified dimension for that matrix. When the subjective specification is not chosen in accordance with data sampling frequency and state dynamic, it leads to inconsistencies in both AD prices and the recovered characteristics for the market. To circumvent this elusive estimation of the AD price matrix, we propose a new and consistent recovery implementation procedure that combines the original insight of the Recovery Theorem with the finite differences of the risk-neutral state dynamic.

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1 Introduction

Recovery is the process of identifying the market's subjective belief and separately, its risk discounting (i.e., risk aversion) from observed prices. The prime challenge in the recovery is the issue of ambiguity. Firm XYZ's stock price is high because market is upbeat about the prospect of its future cash flows, or because market does not mind the risk of these cash flows that much, or a bit of both? Apparently, there can be infinite combinations of belief and risk preference that are consistent with observed prices in the market. It is then difficult to figure out which of these qualified combinations does present the "true" characteristics of the market. In other words, unique recovery seems impossible without further structural or reduced-form restrictions.

While the uniqueness in recovery is impossible in general terms as many counter-examples can quickly disprove the alternative, concentrating on specific terms to rule out ambiguity in recovery is much more interesting investigation. Ross (2013) establishes the Recovery Theorem, which states a set of sufficient conditions for unique recovery. Essentially, unique recovery is achieved when state space structure stays bounded and market's perceived transition probabilities between states stay unchanged from one future period to the next. More specifically, Ross (2013) makes two key innovations, namely formulating the Recovery Theorem and proposing a procedure to implementation the recovery in light of this theorem.

First, the Recovery Theorem shows that physical transition probabilities (that is, market's subjective belief) are simple functions of components of the eigenstate of AD price matrix associated with its largest eigenvalue. The Perron-Frobenius theorem assures the uniqueness of this eigenstate whenever AD price matrix is non-negative and irreducible, resulting in the uniqueness of the recovered belief and risk preference of the market.

Second, the AD price matrix needs to be determined prior to the implementation of the Recovery Theorem. In principle, AD price matrix can be estimated under the stationary premise of the Recovery Theorem. Indeed, assuming no arbitrage and complete market, the current price of a T-period contingent payoff can be computed by rolling forward T

times the same set of one-period contingent (Arrow-Debreu) securities. Putting it other way around, the one-period Arrow-Debreu (AD) price matrix could be effectively backed out from sufficient data on current prices of future payoffs at different maturities T's. The recovery then is implemented by solving for this effective AD matrix's unique eigenstate corresponding to the largest eigenvalue.

The Recovery Theorem is innovative, elegant and mathematically solid. Regardless of the underlying assumptions being strong or weak, the intellectual merit of the theorem is that it stands up facing a known fact that without some sorts of assumptions, unique recovery is impossible. It breeds fresh interests and hopes into a challenging, important and long-standing question. In the current paper we explore instead the aspects of implementing the recovery via solving the AD matrix from prices mentioned above. The main difficulty of this implementation approach lies in the specification of the dimension $n \times n$ for the effective AD matrix to be solved. Specifying this value n is equivalent to consolidating and partitioning the truly continuous state space into n composite and computable states. Hence this state consolidation is both practical and indispensable from computational perspective and limitation. However, we show that two AD price matrices, being solved from the same price data set but under different specifications $n \times n$ and $\overline{n} \times \overline{n}$ respectively, lead generally to two irreconcilable sets of recovered market's belief and risk preference. In other words, while under each specification the respective set of market's recovered characteristics is unique, the sets imply conflicting recovered beliefs, time and risk preferences.

Intuitively, both state dynamic and the available price data sampling frequency ought to contribute to the determination of the consistent state space partition and consolidation. For e.g., when price data is sampled at low frequency (which means certain market's information is overlooked) and state dynamic is volatile, then state space cannot be too densely partitioned because data simply is not informative enough to support such high resolution pricing of state space.[1] Note that while we can prolong time series of data, certain informa-

[1]On top of this theoretical consideration, it is empirically plausible that state dynamic perceived by the market can make the availability of financial contracts and hence their price data vary endogenously with

tion is still missed if sampling frequency stays low, because e.g., certain information may only shows up in the prices in between sampling dates. Our observation hence clearly explains why chopping the state space into finer intervals to make it a better computable (discrete) approximation of the true continuous counterpart does not necessarily solve the consistency issue in the recovery, but likely makes it worse when data sampling and availability cannot be improved as in practice. Because the perceived state dynamic is unknown prior to the recovery process, this consistency issue in the estimation of full AD securities prices is quasi-endogenous to the recovery process. Consequently, below we advocate for the estimation of risk-neutral state dynamic in place of fitting the full but effective AD prices.

More specifically, the recovered physical transition probabilities between two states identifiable (and identical) in two specifications (n and $\overline{n}$) do not converge. Furthermore, arbitrages arise over the pricing of these two sets of AD securities, and hence over the pricing of all other assets built on these two sets. These findings are robust to large n and $\overline{n}$ limits, and to all levels of measurement errors. These findings do not arise because or when the Recovery Theorem's assumptions are marginally or outright violated. In fact, the findings persist even in continuous state space limit, or when price data is both perfect and unlimited, while recovery assumptions are enforced at all time. The indispensable step of state consolidation in solving effective AD matrix may unknowingly involve irreversible loss of information in the price data, such that more and perfect price data does not necessarily correct the estimates. This specification issue can be serious, especially when the analysts implementing the recovery either do not know the "data-generating" specification of the market, or do not have criteria to rank and compare the correctness of different subjective specifications a priori.

Deeper insights into above specification issues in discrete state space can be achieved from attempts to implement recovery in continuous time and state space setting. We first demonstrate that, at any time horizon, the theoretical counterpart of AD matrix in discrete

the state and time horizon. This empirically relevant dependence of data availability on state also needs to be taken into account in the determination of a consistent state space partition and consolidation in the recovery process.

state setting is the infinitesimal generator associated with the risk-neutral state dynamic in continuous state setting. From observable risk-neutral state dynamic, this correspondence allows us to reconstruct (not solve) the AD price matrix at short horizon limit, which is not directly accessible from price data usually available only for longer horizons. The risk-neutral dynamic's estimates indicate the varying density of state variables. Essentially, these estimates serve as a guide to discretizing and consolidating states and time in a harmonious way – similar to an adaptive mesh – that does not create arbitrages. The Recovery Theorem then is implemented in the continuous state and time (i.e., short horizon) limit on a finite difference representation of the infinitesimal operator. Market price of risk, physical state dynamic and physical transition probability density are then uniquely recovered.

Several very recent papers attempt to empirically implement and theoretically generalize the Recovery Theorem of Ross (2013) in several ways. Carr and Yu (2012) demonstrate that unique recovery can be obtained in continuous time and state space if the underlying state variable follows bounded stochastic processes. Their approach imposes appropriate conditions at the boundaries of the state variable's support. They then cast the recovery equation in continuous setting in the form of a Sturm-Liouville boundary value problem, which is known to possess unique positive solution, yielding unambiguous recovery. Carr and Yu (2014) further show that imposing a single condition of Sturm-Liouville type at a single generic interior point of the support would suffice to foster unique positive solution for the recovery. Walden (2013) extends the recovery analysis to unbounded diffusion-type state dynamics and derives necessary and sufficient conditions for unique recovery. Interestingly, these conditions concern only the state dynamics but not the stochastic discount factor. While these results specify constraints on the dynamics about which market's subjective belief can be recovered, they implicate market's risk preferences in a model-free fashion and thus strengthen the recovery paradigm. In an earlier paper (Tran, 2010), we construct a linear differential equation in continuous state space and time, which exactly corresponds to the matrix recovery equation in discrete settings. This correspondence is shown via a mapping between the infinitesimal operator associated with risk-neutral state dynamic and

AD price matrix at short time horizons. Our interest therein is to construct and model the most flexible family of marginal utilities consistent with a given risk-neutral dynamic, and does not result in the unique recovered preference and belief. The current paper extends the mapping mentioned above to any time horizon and also employs finite-difference methodology in the recovery process. Martin and Ross (2013) briefly discuss the practical difficulties in identifying AD price matrix from option prices, specially when the state variable is multi-dimensional. They instead show that AD matrix's largest eigenvalue, which is also the market's time discount factor in the Recovery Theorem, can be identified with the unconditional expected log return on long-maturity bonds. The current paper supports their findings on the elusiveness of AD price matrix by addressing other potential issues in estimating AD price matrix and the subsequent recovery implementation, those arise even in one-dimensional state space. Dubynskiy and Goldstein (2013) emphasize the consequential role and econometric fragility of the imposition of boundary conditions (to obtain unique recovery) on the recovered quantities. By resorting to structural (general equilibrium) restrictions, they show that physical distribution and risk preference can be identified separately and directly from risk-neutral state dynamic and short rate, which in turn can be more reliably estimated from fixed-income price data even when state space is unbounded. The current paper advocates their proposition in estimating and then making explicit use of the risk-neutral state dynamic and short rate. In fact, we employ these estimates to circumvent the direct but subjective identification of AD matrix. Our approach though stays close to the original Recovery Theorem framework by relying less on structural assumptions and more on the boundedness of state space. Huang and Shaliastovich (2013) demonstrate the separate recoverability of market's subjective belief and stochastic discount factor in recursive utility framework. Their approach though requires the knowledge of a specific preference parameter (characterizing the relative importance of preference for the temporal resolution of uncertainty and the inter-temporal elasticity of substitution) and price-dividend ratios of the wealth portfolio.

Our current study is of theoretical nature. We advance our arguments by constructing several thought experiments to enforce perfect and complete price data inputs to assume away measurement error issues and to focus on specification aspects of recovery implementation. Our illustrating examples are on data simulated under thought-experiment environment. A subsequent work will address the empirical aspects of specification issues and our proposed implementation procedure for the recovery.

The current paper is structured as follows. Section 2 briefly re-derives the Recovery Theorem and describes in details its implementation procedure via solving first an effective AD price matrix. Section 3 discusses various issues arising from subjectively specifying the dimension of the AD price matrix. We derive formal sufficient-and-necessary condition for different specifications to produce consistent recovered belief and risk preference. This condition is highly restrictive, thus formally validates our findings on specification issues. Section 4 proposes a different approach to implement the recovery via risk-neutral state dynamic estimation and finite differencing methodology. Section 5 concludes. Appendices present technical proofs and other quantitative results supporting the Recovery Theorem and its implementation.

2 Discrete state-time setting

Discrete space-time setting is the original and also simple (natural) environment to present the ideas and details underlying the Recovery Theorem due in technical part to the power and beauty of the associated apparatus of matrix (linear) algebra. This setting is also the most suitable for computerized implementation of the theorem. But as we will see, discrete space-time setting also gives rise to the potential challenges in the specification of the recovery process.

2.1 Recent advances in the recovery

We first briefly re-count Ross (2013)'s two key ideas to recovery. We also discuss several recent results from related literature. This sets the stage for our subsequent study of specification aspects of recovery.

Idea 1: The Recovery Theorem

Let us consider the pricing of a uncertain future payoff z_{t+1}. We assume **(i)** no-arbitrage and **(ii)** complete financial market and hence, there exists unique stochastic discount factor (SDF) in the economy. Note that the SDF can be cast as the marginal utility of economy's representative agent.[2] The current price $P_{z,t}$ can be computed in either risk-neutral measure Q or physical measure P,

$$P_{z,t} = E_t^Q \left[e^{-r_t} z_{t+1} \right] = E_t^P \left[\delta \frac{M_{t+1}}{M_t} z_{t+1} \right], \tag{1}$$

where r_t is the risk-free rate (short rate, for the period from t to $t+1$), δ is time discount factor, and M_t is SDF (or marginal utility). In the special case when the payoff stream is chosen such that,

$$z_t \equiv \frac{1}{M_t}, \tag{2}$$

the above pricing equation becomes,

$$E_t^Q \left[e^{-r_t} \frac{1}{M_{t+1}} \right] = \delta \frac{1}{M_t} \qquad \text{or,} \qquad \sum_{j=1}^{n} Prob^Q(i,t;j,t+1) e^{-r_{i,t}} z_{j,t+1} = \delta z_{i,t},$$

where $\{i,t\}$, and $\{j,t+1\}$ denote explicitly the current and future state/time, $Prob^Q$ the risk-neutral transition probability, and z the inverse of stochastic discount factor (2). Under further key assumptions that **(iii)** discrete state space is finite, $n < \infty$ and **(iv)** state dynamic

[2] The role of representative agent is not essential to obtain uniquely the key decomposition in the recovery as explained by Martin and Ross (2013).

is Markovian[3], the above equation becomes,

$$\sum_{j=1}^{n} Prob^{Q}(i,j)e^{-r_i}z_j = \delta z_i.$$

On the other hand, given current state i, the price of j-th one-period Arrow-Debreu (AD) security offering unit payoff only in future state j is

$$A_{ij} = E_i^Q\left[e^{-r_i}\mathbb{1}_j\right] = \sum_{k}^{n} Prob^{Q}(i,k)e^{-r_i}\mathbb{1}_{k,j} = Prob^{Q}(i,j)e^{-r_i}.$$

Combining the last two equations we arrive at the fundamental equation underlying the recovery process

$$\sum_{j=1}^{n} A_{ij}z_j = \delta z_i,\ \forall i \Longleftrightarrow \begin{pmatrix} A_{11} & \cdots & A_{1n} \\ \vdots & \ddots & \vdots \\ A_{n1} & \cdots & A_{nn} \end{pmatrix} \begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix} = \delta \begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix} \Longleftrightarrow Az = \delta z, \qquad (3)$$

where in (3), the last two equations simply are matrix forms of the first; A is $n \times n$ matrix of one-period AD prices, z is $n \times 1$ vector of SDF reciprocals (2). In particular, no-arbitrage requires that SDF in all states, and hence all elements of z, be strictly positive. Equation (3) succinctly asserts that the time discount factor δ and the inverse of SDF is an eigenpair of AD price matrix. The change from risk-neutral (Q) to physical (P) measure[4] then equally succinctly implies the physical distribution (that is, the transition probability under market's

[3]In particular, this Markovian assumption implies that the stochastic discount factor M_t is function only of the state at t, but not the path leading to that state.

[4] Change of measure is characterized by the unique Radon-Nikodym derivative ξ, which arises from the pricing (1),

$$\sum_{j=1}^{N} Prob^{Q}(i,j)e^{-r_i}z_j = \sum_{j=1}^{N} Prob^{P}(i,j)\frac{M_j}{M_i}z_j,\ \forall z \Longrightarrow \xi \equiv \frac{Prob^{P}(i,j)}{Prob^{Q}(i,j)} = \frac{e^{-r_i}M_i}{\delta M_j}.$$

subjective belief) from this same eigen-problem,

$$\frac{Prob^P(i,j)}{Prob^Q(i,j)} = \frac{e^{-r_i} M_i}{\delta M_j} = \frac{e^{-r_i} z_j}{\delta z_i} \Rightarrow Prob^P(i,j) = \frac{Prob^Q(i,j) e^{-r_i} z_j}{\delta z_i} = \frac{A_{ij} z_j}{\delta z_i}. \tag{4}$$

From this also follows the physical transition probability for any horizons. For e.g., for 2–period transitions respectively (hereafter, A^t_{ij} always denotes the $ij-$ entry of the power matrix A^t),

$$Prob^{P,2}(i,j) = \sum_k Prob^P(i,k) Prob^P(k,j) = \sum_k \frac{A_{ik} z_k}{\delta z_i} \frac{A_{kj} z_j}{\delta z_k} = \frac{A^2_{ij} z_j}{\delta z_i},$$

and then t–period transitions[5]

$$Prob^{P,t}(i,j) = \frac{A^t_{ij} z_j}{\delta^t z_i}.$$

When the transition probability from any initial state at t to any state at $t+\tau$ ($\mathbb{N} \ni \tau \geq 1$) is strictly positive, by no-arbitrage the associated contingent prices, i.e., all elements of matrix $A^{t+\tau}$, are as well strictly positive. The Frobenius' extension of Perron's theorem (see Appendix) of linear algebra then assures that one-period AD matrix A (as well as $t+\tau$-period AD matrix $A^{t+\tau}$) has *unique* eigenstate z of all strictly positive elements, associated with a strictly positive eigenvalue δ.

Conceptually, it is this uniqueness of the SDF strictly positive in all states, $M_i = \frac{1}{z_i}$, strictly positive time discount factor δ, and consequently the uniqueness of the physical distribution $Prob^P(i,j)$ in (4) that is central to Ross's (2013) recovery. Formally, his Recovery Theorem establishes various assumptions **(i)-(iv)** as sufficient conditions for the unique recovery of physical distribution (or market's subjective belief) P, stochastic discount factor (or market's risk adjustment) M, and time discount factor δ.

[5]In the long horizon limit (see Martin and Ross (2013) for an application of long horizon limit on bond pricing), $\lim_{t\to\infty} \frac{A^t}{\delta^t} = z \times w^T$, where $1 \times n$ vector w^T satisfying $w^T A = \delta w^T$ is the left eigenstate associated with the largest eigenvalue δ. Then $\lim_{t\to\infty} Prob^{P,t}(i,j) = (z \times w^T)_{ij} \frac{z_j}{z_i} = z_i w_j \frac{z_j}{z_i} = w_j z_j$ which is independent of initial state i. In other words, long horizon limit yields a stationary distribution of target state j. This stationary distribution is a proper distribution, because $\sum_j w_j z_j = w^T \times z = 1$ by requisite normalization for the limiting decomposition $\lim_{t\to\infty} \frac{A^t}{\delta^t} = z \times w^T$ to work.

Idea 2: Recovery implementation via AD securities

On the basis of the close relation between AD prices and SDF discussed above, it is natural to implement the recovery by solving the the eigen-problem (4). However, full set of one-period AD securities and their prices, especially the securities $A_{i'j}$ contracted for initial states i' different from the current state i at time t, are not directly available in the market at current time t. A key advantage of the Recovery Theorem is that it would employ price data of only *forward-looking* contracts. As a result, at any time, the Recovery Theorem aims to back out uniquely the market's forward-looking belief about future prospect of the economy at that point in time, while allowing market's belief to change over time.[6] In contrast, other recovery methods making use of the historical data most likely need to rely on some sort of stronger assumption that market's belief in the past and future be either identical or structurally related.

Under the very same assumptions of the Recovery Theorem, [7] full set of one-period AD prices may be implied from financial assets contracted only for current initial state i but having longer maturities.

To understand this approach, we start with the seemingly obvious matrix power identity $A^{t+1} = A^t A$. To maintain no-arbitrage, it is necessary that elements of power matrix A^t be prices of $t-$period AD securities for all t: $\mathbb{N} \ni t \geq 1$.[8] For each integer $t \geq 1$, we collect i-th

[6]For the Recovery Theorem, the respective assumption is just that, at any time t, market *believes* that the economy has some specific and *fixed* Markovian dynamic into future of all lengths. But as time passes, at any ulterior time $t+\tau$, the Recovery Theorem allows market to have a possibly different specific and fixed Markovian dynamic in its belief.

[7]Specifically, these are no-arbitrage, complete market and Markovian state dynamic

[8]That is, $A^t_{i,j}$ is the current price of the contract paying 1 only if the future state in t periods from now is j, given the current state being i.

row of this identity, (with i denoting the current state of the economy),

$$\begin{aligned}
(A^2_{i,1}, \ldots, A^2_{i,n}) &= (A_{i,1}, \ldots, A_{i,n}) \times A, \\
(A^3_{i,1}, \ldots, A^3_{i,n}) &= (A^2_{i,1}, \ldots, A^2_{i,n}) \times A, \\
&\vdots \\
(A^{t+1}_{i,1}, \ldots, A^{t+1}_{i,n}) &= (A^t_{i,1}, \ldots, A^t_{i,n}) \times A, \\
&\vdots
\end{aligned} \tag{5}$$

or in matrix form after stacking,

$$\begin{pmatrix} A^2_{i,1} & \ldots & A^2_{i,n} \\ A^3_{i,1} & \ldots & A^3_{i,n} \\ \vdots & \ldots & \vdots \\ A^{t+1}_{i,1} & \ldots & A^{t+1}_{i,n} \\ \vdots & \ldots & \vdots \end{pmatrix} = \begin{pmatrix} A_{i,1} & \ldots & A_{i,n} \\ A^2_{i,1} & \ldots & A^2_{i,n} \\ \vdots & \ldots & \vdots \\ A^t_{i,1} & \ldots & A^t_{i,n} \\ \vdots & \ldots & \vdots \end{pmatrix} \times \begin{pmatrix} A_{1,1} & \ldots & A_{1,n} \\ \vdots & \ddots & \vdots \\ A_{n,1} & \ldots & A_{n,n} \end{pmatrix} \tag{6}$$

$$\text{or,} \quad \mathcal{A}_{2\uparrow} = \mathcal{A}_{1\uparrow} \times A$$

This is a system of linear equations determining *all* one-period AD prices $\{A_{k,j}\}$, $\forall k, j$, from prices of longer-maturity AD prices $A^t_{i,j}$, $\forall t, j$ contracted *only* on the current state i. The no-arbitrage relation (6) is central to Ross (2013)'s proposed procedure to recovery, which consists of *four* steps. *First*, we collect price data $\{A^t_{i,j}\}_{j,t}$ for *forward-looking* contracts available today (that is, written on the current state i of today). *Second*, we obtain the one-period AD price matrix A from inverting system (6). Note that this is possibly an open-ended system, as set of maturities $\{t\}$ is limited only by the availability of data. This observation leads to two approaches, namely the just-identified and ordinary least squares (OLS), to determine the one-period AD price matrix A.

- **Just-identified** determination of AD price matrix: In this approach, we collect precisely $n+1$ consecutive data maturities (e.g. $t = 1$ to $n+1$), which are just enough to solve for the $n \times n$ matrix A by inverting (6),

$$A \equiv \begin{pmatrix} A_{1,1} & \dots & A_{1,n} \\ \vdots & \ddots & \vdots \\ A_{n,1} & \dots & A_{n,n} \end{pmatrix} = \begin{pmatrix} A_{i,1} & \dots & A_{i,n} \\ \vdots & \ddots & \vdots \\ A^n_{i,1} & \dots & A^n_{i,n} \end{pmatrix}^{-1} \times \begin{pmatrix} A^2_{i,1} & \dots & A^2_{i,n} \\ \vdots & \ddots & \vdots \\ A^{n+1}_{i,1} & \dots & A^{n+1}_{i,n} \end{pmatrix}. \quad (7)$$

A subtle question here is that which $n + 1$ consecutive maturities should one employ in the just-identified determination of AD price matrix A? We show in equation (14) below that this choice of maturities matters whenever the specification n of the system is unknown to the analysts who implement the recovery.

- **OLS** determination of AD price matrix: In this approach, we employ all available data which renders system (6) over-identified. Accordingly, we perform a best-fit determination of one-period AD prices which minimize the squared errors of the no-arbitrage pricing equations in (5) and (6). The best fit yields a familiar OLS estimate for AD price matrix,

$$A = \left(\mathcal{A}^T_{1\uparrow}\mathcal{A}_{1\uparrow}\right)^{-1} \mathcal{A}^T_{1\uparrow}\mathcal{A}_{2\uparrow}. \quad (8)$$

where superscript T denotes transpose operation on matrices. Naturally, the OLS estimation would be most useful when price data is imperfect (sparse, fragmented or with noises). We will explicitly analyze the OLS determination for AD price matrix while establishing related results in Proposition 2 below.

Third, we solve equation (3) for the largest eigenvalue δ and the associated eigenstate z, which are all-positive by virtue of Perron-Frobenius theorem. *Fourth*, we recover the marginal utility M_t from equation (2), and the physical distribution $Prob^P$ from equation (4).

A practical and important issue remains; in case we do not know the correct (or "most appropriate") number of states, how a particular specification n of choice for the number of states would affect the quality and reliability of recovered quantities? Even when ultimate specification of the world is the continuous state space, computerized estimation procedures work only with discrete (or discretized) states and the specification issue remains relevant. Furthermore, in section 4 we show that higher number of states n does not necessarily provide better approximation for the underlying continuous state space setting. Before concluding this section, for an overview, we examine some special cases proposed in the recent literature, in which recovery can be achieved without some tenets of the Recovery Theorem.

A digression: Recovery without Perron-Frobenius theorem

We note that unique recovery can be achieved without the marvel of the Perron-Frobenius theorem in special cases. Ross (2013), and Huang and Shaliastovich (2013) report one such case. In a representative-agent framework and when consumption growth depends only on the destination state j (but not original state i), the marginal utility ratio is contingent only on j. That is, $\delta\frac{M_{j,t+1}}{M_{i,t}}$, or equivalently $\delta\frac{z_{i,t}}{z_{j,t+1}}$ is some state-contingent entry, say m_j. From equation (3) then follow mechanically and unambiguously these marginal utility ratios m_j's (i.e., preferences)

$$\sum_j A_{ij}\frac{1}{m_j} = 1,\ \ \forall i \Longrightarrow \begin{pmatrix} \frac{1}{m_1} \\ \vdots \\ \frac{1}{m_n} \end{pmatrix} = \begin{pmatrix} A_{11} & \cdots & A_{1n} \\ \vdots & \ddots & \vdots \\ A_{n1} & \cdots & A_{nn} \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix},$$

and the physical distribution (see also (4)), $Prob^P(i,j) = \frac{A_{ij}z_j}{\delta z_i} = A_{ij}\frac{1}{m_j}$. without invoking the Perron-Frobenius theorem. It turns out that, in this special case, instead of n^2 (unknown) combinations $\{\delta\frac{z_i}{z_j}\}$ with (unknown) discount factor δ, one works with only n (unknown) entries $\{m_j\}$. While being simple, these spacial cases mask the characteristic prowess of the Recovery Theorem, which is the *unique* resolution of *more* unknowns from *less* available pricing equations.[9]

[9] This is technically a consequence of the Perron-Frobenius eigen-problem, in which the eigenvalue, or time discount factor, is endogenously identified.

3 State space specification

In this section we study issues related to the specification of number of states n within discrete state space framework. Specification issues in continuous state space setting are considered in section 4.

3.1 Consistency in recovery

Because *a priori* we do not know the correct (or "most appropriate") number n of states. Given a specific value n,[10], the Recovery Theorem yields a unique physical distribution of n-dimensional state space corresponding to this particular value n. Another specific choice of $\overline{n}$ yields another unique corresponding physical distribution. Then a natural question arises that whether physical distributions – each obtained uniquely by virtue of the Recovery Theorem for a specification n – are *consistent* with one another?

To answer this question we first discuss in what sense the distributions recovered in different specifications need to be consistent with one another. We naturally assume that different specifications cover the same support of state space.[11] One specification partitions this support into n states, a second partitions it into $\overline{n}$ states. Denote these partitions Ω and $\overline{\Omega}$, and the associated recovered physical distributions $Prob^P$ and $\overline{Prob}^P$, respectively. Let F be a generic subset of this support, and hence F is common to both specifications. At the minimum, the consistency between these recovered distributions stipulates the following requirements.

1. The recovered *aggregate* probability of the transition from current state i today (at t) to any state in F be the same next period (at $t+1$) and independent of the specification,

[10] And that all assumptions underlying the Recovery Theorem hold.

[11] This support is the entire "true" state space when the latter is known, or any well-defined subset of the unknown "true" state space.

as long as F is as common to both partitions, [12]

$$\sum_{j\in F} Prob^P(i,j) = \sum_{\bar{j}\in F} \overline{Prob}^P(i,\bar{j}), \quad i \equiv \text{today's state}, \forall F \subset \left(\Omega \cap \overline{\Omega}\right).$$

2. More stringently, the consistency of the recovery apply to *all* transitions that the recovery procedure recovers, but *not only* the transitions starting from the current state i of the world. That is, when the state partitions Ω and $\overline{\Omega}$ of two different specifications share some/any common initial state i and some/any common subset F of final states, the from–i–to–F transition probabilities recovered using two specifications need be the identical (up to numerical errors),

$$\sum_{j\in F} Prob^P(i,j) = \sum_{\bar{j}\in F} \overline{Prob}^P(i,\bar{j}), \quad \forall i, F \subset \left(\Omega \cap \overline{\Omega}\right).$$

3. The recovered time discount factor, i.e. the largest eigenvalue in the fundamental eigen-problem equation (3) of the Recovery Theorem, be identical and independent of the recovery specifications,

$$\delta = \bar{\delta}.$$

4. The recovered stochastic discount factors associated with some/any state pair i, j (which is common to different specifications) be identical and independent of the recovery specifications,[13]

$$\frac{M_{j,T}}{M_{i,t}} = \frac{\overline{M}_{j,T}}{\overline{M}_{i,t}}, \ \ \forall i, j \in \left(\Omega \cap \overline{\Omega}\right).$$

[12]The equality needs to hold up to numerical errors, which results from the numerical implementation of the recovery.

[13]This requirement concerns possibly an intricate concept intrinsic to discrete-state setting of consolidating several sub-states into a combined state, and how such operations shape the interpretation of the associated state-contingent SDF, especially when the SDF is interpreted as the representative agent's marginal utility. We defer the discussion relevant to this issue until after Proposition 1, and also to section 4.

To illustrate the consistency in recovery, the diagram in Figure 1 sketches representatively two recovery specifications of $n = 3$ and $\overline{n} = 2$ respectively. For simplicity, the state space

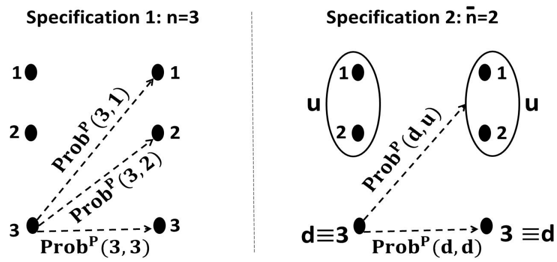


Figure 1: Different recovery specifications, with two- and three-dimensional state space partitions.

partitions $\Omega \equiv \{1,2,3\}$ and $\overline{\Omega} \equiv \{u, d\}$ are chosen to have identical support, and

$$\{1,2\} = \{u\}; \qquad\qquad \{3\} = \{d\}.$$

Let us also assume that the current state is the common state $\{3\}$ in $n = 3$ specification (or equivalently, $\{d\}$ in $\overline{n} = 2$ specification). Concerning the recovered physical distributions, typical consistency requirements read

$$\overline{Prob}^P(d,u) = Prob^P(3,1) + Prob^P(3,2); \qquad\qquad \overline{Prob}^P(d,d) = Prob^P(3,3).$$

3.2 Specification versus measurement errors: a thought experiment

It is important to note that the specification issue in the recovery is distinct from the measurement error topic. Typically, the measurement errors arise from insufficient, low-quality or conflicting data that may also exclude e.g., extreme but unlikely events (rare disasters). Our current study instead focuses upon the consistency among different specifications of the implementation of Recovery Theorem. As we show below, it turns out that the consistency issue is relevant even when data and data processing power are perfect, and thus even when measurement errors are absent. It is then desirable that, for our current analysis, the consistency and measurement error aspects of the recovery implementation be decoupled from one another before insights into their distinct effects on recovered results are gained. To this end, we consider the following thought experiment that employs presumably perfect price data, and thus abstracts away from all possible measurement errors, leaving the specification and its consistency the only force at work. Note that below, for ease of exposition, we idealistically often refer to one specification as "true" (i.e., data-generating), and the second as "subjective". Our arguments and findings though hold for any two specifications that are not the same.

The **thought experiment** consists of the following steps.

Step 0–Market structure: Perfect data would imply an n–state market, the associated $n \times n$ AD price matrix A (of one-period contracts), market's time discount factor δ, physical distribution (i.e., market belief) $Prob^P(i,j)$, SDF (i.e., market's marginal utility) $M_k = \frac{1}{z_k}$, $\forall k \in \{1, 2, \ldots n\}$, in line with the Recovery Theorem (3); $Az = \delta z$, $Prob^P(i,j) = \frac{A_{i,j} z_j}{\delta z_i}$.

Step 1–Generating perfect data: We start with an $n \times n$ AD price matrix A having all positive entries, and having sum of entries in each (and any) row smaller than one.[14] The perfect price data that produces A are reconstructed in our thought experiment recursively according to system (5), starting from i-th row of A (that is, the prices of all n one-period AD securities contracted on the current state i),

$$(A_{i,1}^{t+1}, \ldots, A_{i,n}^{t+1}) = (A_{i,1}^{t}, \ldots, A_{i,n}^{t}) \times A, \quad \forall t = 1, 2, 3 \ldots . \tag{9}$$

[14] Given current state i, row sum $\sum_j A_{ij}$ is the price of risk-free bond paying surely \$1 next period, these sums are less than one to enforce no-arbitrage and positive risk-free rate.

Hereafter, we assume that we are provided with this perfect and unlimited [15] price data set, referred to as original or raw data below (the notation follows from (6)),

$$\mathcal{A}_{1\uparrow} \equiv \{A^t_{i,j} : \forall t \geq 1, \forall j\}, \tag{10}$$

associated with some *fixed* initial state i, *without knowledge* of the data-generating but *hidden* specification n of the market as well as the original $n \times n$ AD price matrix A. The restrictions aim to mimic the realistic situation faced by market analysts implementing the recovery process on a generic day, except that in the current thought experiment data is unlimited and perfect.

Step 2–Specification anew: Pondering upon a wealth of perfect data set $\mathcal{A}_{1\uparrow}$ (10), but not knowing the underlying data-generating specification n, analysts attempt to recover an $\overline{n}$–state market and its belief, time and risk preferences uniquely.[16] We consider only the more interesting situation in which the analysts ponder upon a lower-dimensional recovery specification than that of original data[17]

$$\overline{n} < n. \tag{11}$$

Accordingly, we hereafter refer to the specification $\overline{n}$ tumbled on by analysts as the *consolidated* system, and use an "upper bar" to denote all quantities associated with this system. The recovery procedure on consolidated data will follow the sequence outlined above, from solving the one-period $\overline{n} \times \overline{n}$ AD price matrix (from data in $\mathcal{A}_{1\uparrow}$) to solving the resulting eigen-problem (3). This is equivalent to unknowingly consolidating n–state data system into

[15] Data set $\mathcal{A}_{1\uparrow}$ in (10) is unlimited as it can be generated for any maturities $t \geq 1$ via recursive equation (9).

[16] The choice of specification $\overline{n}$ might be suggested/dictated by computing capacity of the analysts.

[17] The alternative situation $\overline{n} > n$ is not in line with the irreducibility assumption of the Recovery Theorem, the violation of which will render the recovery ambiguous (non-unique). Furthermore, when original data is perfect as in the current thought experiment, if analysts tumble on a subjective specification of higher dimension than what generates the data ($\overline{n} > n$), they can quickly discover the trivial replication of states. This situation happens, for a visualization, when a single data-generating (or "true") state i is split into two artificial and identical states i_1, i_2 by repetition.

$\overline{n}$ states, which is both a proper and prevalent arbitrage-free routine. It is as natural as, say, the contract paying when a reference index next period falls in the range $(2, 5]$ costs as much as one contract paying in $(2, 3]$ together with another contract paying in $(3, 5]$ next period.[18] This step of data consolidation is indispensable from practical computational perspectives, and is often carried out in conjunction with the Breeden-Litzenberger formula (Appendix E) to fit the volatility surface.

In the recovery setting, for a concrete illustration, the unknowing consolidation of states k and $k+1$ of the original n–state system results in the consolidated price data in the after-stacking form

$$\mathcal{A}_{1\uparrow} \equiv \begin{pmatrix} A_{i,1} & \dots & A_{i,k} & A_{i,k+1} & \dots & A_{i,n} \\ \vdots & \dots & \vdots & \vdots & \dots & \vdots \\ A^h_{i,1} & \dots & A^h_{i,k} & A^h_{i,k+1} & \dots & A^h_{i,n} \\ \vdots & \dots & \vdots & \vdots & \dots & \vdots \end{pmatrix} \longrightarrow$$

$$\overline{\mathcal{A}}_{1\uparrow} \equiv \begin{pmatrix} A_{i,1} & \dots & A_{i,k} + A_{i,k+1} & \dots & A_{i,n} \\ \vdots & \dots & \vdots & \dots & \vdots \\ A^h_{i,1} & \dots & A^h_{i,k} + A^h_{i,k+1} & \dots & A^h_{i,n} \\ \vdots & \dots & \vdots + \vdots & \dots & \vdots \end{pmatrix} \equiv \begin{pmatrix} \overline{A}_{i,1} & \dots & \overline{A}_{i,\overline{n}} \\ \vdots & \dots & \vdots \\ \overline{A}^h_{i,1} & \dots & \overline{A}^h_{i,\overline{n}} \\ \vdots & \dots & \vdots \end{pmatrix} \tag{12}$$

Note that because analysts tumble on the specification $\overline{n}$, they consolidate and then work with $\overline{n}$–state consolidated data set $\overline{\mathcal{A}}_{1\uparrow}$, but not the raw data set $\mathcal{A}_{1\uparrow}$. The associated $\overline{n} \times \overline{n}$ AD price matrix $\overline{A}$ is solved from consolidated data recursively by solving a linear system

[18] In term of consolidating the final states, the price of contract paying in either final states j or k is sum of of prices of two contracts; one pays only in final state j, the other only in state k.

similar to (6),

$$\begin{pmatrix} \overline{A}^2_{i,1} & \dots & \overline{A}^2_{i,\overline{n}} \\ \overline{A}^3_{i,1} & \dots & \overline{A}^3_{i,\overline{n}} \\ \vdots & \dots & \vdots \\ \overline{A}^{t+1}_{i,1} & \dots & \overline{A}^{t+1}_{i,\overline{n}} \\ \vdots & \dots & \vdots \end{pmatrix} = \begin{pmatrix} \overline{A}_{i,1} & \dots & \overline{A}_{i,\overline{n}} \\ \overline{A}^2_{i,1} & \dots & \overline{A}^2_{i,\overline{n}} \\ \vdots & \dots & \vdots \\ \overline{A}^{t}_{i,1} & \dots & \overline{A}^{t}_{i,\overline{n}} \\ \vdots & \dots & \vdots \end{pmatrix} \times \begin{pmatrix} \overline{A}_{1,1} & \dots & \overline{A}_{1,\overline{n}} \\ \vdots & \ddots & \vdots \\ \overline{A}_{\overline{n},1} & \dots & \overline{A}_{\overline{n},\overline{n}} \end{pmatrix} \tag{13}$$

$$\text{or,} \quad \overline{\mathcal{A}}_{2\uparrow} = \overline{\mathcal{A}}_{1\uparrow} \times \overline{A}$$

The consolidated AD price matrix $\overline{A}$ is then obtained by either just-identified estimation (7) or OLS (8), on consolidated data set $\overline{\mathcal{A}}_{1\uparrow}$.

Step 3–Recovery on $\overline{n}$–specification: Finally, the recovery is implemented on the consolidated data set $\overline{\mathcal{A}}_{1\uparrow}$ in (12) along the line of eigen-problem (3), and (4),

$$\overline{A}\overline{z} = \overline{\delta}\overline{z}, \qquad \overline{Prob}^P(i,j) = \frac{\overline{A}_{i,j}\overline{z}_j}{\overline{\delta}\overline{z}_i}, \quad \forall i,j \in \{1,\dots,\overline{n}\}.$$

This thought experiment allows us to isolate and study the role and effects of the specification on the recovery process in discrete state space setting. We first describe and discuss these effects qualitatively via numerical examples, before establishing more formal and generic results.

3.3 Specification issues: Informal discussion

Specification subjectively chosen by analysts might lead to unknowingly missing out some useful information encoded in price data, irreversible loss of information, (spurious) arbitrage generation, and possible inconsistency in recovered distributions. While our explicit examples below concern small values n, $\overline{n}$ for state space dimensions, the convergence issues persist for larger values. Indeed, in section 4 we show that the divergence may arise even in continuous state space limit.

Information miss-out: When analysts employ the just-identified approach to determine the AD price matrix, they would overlook important information in the data. Indeed, it is sufficient to to work with any (linearly independent) $\overline{n}$ rows from the consolidated system (13) to just-identify the $\overline{n} \times \overline{n}$ matrix $\overline{A}$. This is in contrast to the "true" nature of the raw data (9)–(10), which was constructed (in the thought experiment) in such a way that the "true" AD matrix A can only be adequately exposed when n rows from the raw data system are used. Under assumption (11), this clearly shows that analysts are (unknowingly) content with employing less data than "truly" needed in their just-identified estimation of the AD price matrix. An implication then is that implementing recovery on subjective specification would yield results different from those of the "true" but hidden setting of dimension n.

Given that this information miss-out is intrinsic to the just-identified estimate of the AD price matrix, it only seems natural to commit to the more practical OLS estimation approach which employs all available data as inputs. However, as Proposition 2 shows, even unlimited and perfect price data does not correct the bias in the OLS estimate of the subjective specification.

Finally, we note that this type of information miss-out matters only in the case $\overline{n} < n$ (11). When subjective specification coincides with the one that generates the price data ($\overline{n} = n$), employing any $n + 1$ consecutive data maturities (or more) gives the identical estimate for AD price matrix A by virtue of the *perfect* n–state data structure assumed in this thought experiment.

Information loss: Consolidating price data may also give rise to another type of informational inefficiency. We note that different worlds of raw data sets may result in identical consolidated set. Consequently, while implementing recovery on the consolidated data gives uniquely recovered belief, time and risk preferences, these characteristics are not necessarily those of the market. A numerical example illustrates this point emphatically. Let us consider two distinct raw data sets, both of "true" $n = 5$ states. Because this "true" (i.e., data-generating) specification is unknown to analysts, they attempt to recovery instead a

$\overline{n} = 3$–state system.[19] Accordingly, analysts solve either of the following linear equation systems on the consolidated data of the form (12)–(13), depending on which world of raw data they are provided with (recall that summing of columns corresponds to states' consolidating),

$$\begin{aligned} &\text{World I:} \quad \begin{pmatrix} 1 & 1+3 & 4+3 \\ 1 & 2+3 & 5+4 \\ 1 & 3+3 & 6+5 \end{pmatrix} = \begin{pmatrix} 1 & 1+2 & 3+3 \\ 1 & 1+3 & 4+3 \\ 1 & 2+3 & 5+4 \end{pmatrix} \times \overline{A}_I; \\ &\text{World II:} \quad \begin{pmatrix} 1 & 2+2 & 5+2 \\ 1 & 1+4 & 8+1 \\ 1 & 5+1 & 4+7 \end{pmatrix} = \begin{pmatrix} 1 & 2+1 & 1+5 \\ 1 & 2+2 & 5+2 \\ 1 & 1+4 & 8+1 \end{pmatrix} \times \overline{A}_{II}; \end{aligned} \tag{14}$$

Clearly, two systems above are numerically identical. This implies that the analysts obtain indistinguishable AD price matrices ($\overline{A}_I = \overline{A}_{II}$), and consequently recover identical sets of market's characteristics (belief, time and risk preferences) if they tumble on a subjective specification of $\overline{n} = 3$. Yet the two original worlds (corresponding to raw price data before we sum columns 2 with 3, and 4 with 5) are distinguishable, which are associated with two completely different original ("true", or data-generating) sets of underlying market's characteristics. The consolidation step in implementing recovery may act to mask these differences. This many–to–one mapping between original and consolidated data implies that information may be unknowingly and irreversibly lost, and the recovery may be distorted by specification issues.

Spurious arbitrage: The indispensable step of consolidating data in the recovery implementation may also result in arbitrages in pricing $\overline{n} \times \overline{n}$ AD matrix in the subjective specification $\overline{n}$, even though original price data is arbitrage-free. Again, let us first consider a numerical example before discussing its causes. Suppose emphatically that the consolidation on the original $n = 3$-state space leads to $\overline{n} = 2$-state system. The correspondence between original

[19]For the sake of specificity, in this numerical example, we assume that the bundling of $n = 5$ original states $\{1, 2, 3, 4, 5\}$ into $\overline{n} = 3$ consolidated states $\{u, m, d\}$ is as follows; $\{u\} = \{1\}$, $\{m\} = \{2, 3\}$, $\{d\} = \{4, 5\}$.

and consolidated states is just as depicted in Figure 1. The original 3×3 AD price matrix A is given below, from which we can solve for the consolidated 2×2 AD price matrix $\overline{A}$ (by following steps (12)–(13)),

$$A = \begin{pmatrix} 0.0759 & 0.5711 & 0.1059 \\ 0.1226 & 0.2094 & 0.6222 \\ 0.6496 & 0.1986 & 0.0825 \end{pmatrix} \Longrightarrow \overline{A} = \begin{pmatrix} 0.2664 & 0.1621 \\ 1.9907 & -0.1999 \end{pmatrix} \tag{15}$$

A negative AD price of $\overline{A}_{22} = -0.1999$ implies an arbitrage opportunity[20] in the consolidated system. This is an arbitrage spuriously created by subjective identification scheme. On one hand it prompts the analysts to change their identification, though incorrect subjective identifications are not always revealed by some resulted negative AD prices as we seen.[21] On the other hand, this mere possibility of arbitrage simply indicates big-picture potential issue of subjective identification.

Discussion: Our thought experiment is designed also to demonstrate potential arbitrages arising in any implementation of the recovery which involves consolidation of states. To shed light into their causes within discrete state space setting,[22] let us closely reexamine the consolidation steps (12)–(13). We recall that the recovery on the original data involves solving for AD matrix A from the linear system (6) in stacked form, $\mathcal{A}_{2\uparrow} = \mathcal{A}_{1\uparrow} \times A$. In this matrix form, summing columns of matrix $\mathcal{A}_{2\uparrow}$ (i.e., consolidating final states) is rigorous *only* when being accompanied by simultaneously summing similar columns of matrix A, leaving matrix $\mathcal{A}_{1\uparrow}$ intact. Instead, the recovery on subjective specification (13) employs exclusively consolidated data in both sides of this (stacked) linear system. This is equivalent to summing columns $\mathcal{A}_{1\uparrow}$ (into $\overline{\mathcal{A}}_{1\uparrow}$) accompanied by simultaneously summing similar columns of $\mathcal{A}_{2\uparrow}$

[20] That is, suppose the current state is d in consolidated system (or state 3 in the original system) in Figure 1, an investor is paid \$0.1999 at time t if he agrees to hold a AD security which pays \$1 if state at $t+1$ is d, and 0 in all other states.

[21] Incorrect subjective identifications may only result in information miss-out or/and information loss, but not negative AD prices.

[22] The discussion on causes of spurious arbitrages within continuous state space setting is deferred until section 4.

(into $\overline{\mathcal{A}}_{2\uparrow}$). Subsequently, the consolidated AD matrix is endogenously established to make $\overline{\mathcal{A}}_{2\uparrow} = \overline{\mathcal{A}}_{1\uparrow} \times \overline{A}$ in (13) an equality. Though such $\overline{A}$ can be found to enforce this equality *technically*, this discussion points out that the $\overline{n} \times \overline{n}$ consolidated $\overline{A}$ determined in this way is not entirely *economically* linked to the original $n \times n$ AD price matrix A. This disconnection can give rise to the arbitrages in the former (consolidated) AD securities prices in $\overline{A}$, even when they are absent in the original system A.

Consistency in recovered beliefs and time preferences: The disconnection mentioned above between original and consolidated AD price matrices may also give rise to the mismatches in the recovered time discount factor and recovered physical distributions (beliefs). For an illustration, we randomly generate $N = 30$ original matrices $n \times n$ AD price matrices A, $n = 3$. Each matrix has all positive entries, and sum of entries in any row smaller than one (see footnote 14). Then $N = 30$ corresponding consolidated matrices $\overline{n} \times \overline{n}$ matrices $\overline{A}$, $\overline{n} = 2$ are determined in procedure (12)-(13), and described schematically in Figure 1. The time discount factors δ and $\overline{\delta}$ are recovered respectively as largest eigenvalues of A and $\overline{A}$, accordingly 3. The absolute value of their difference is plotted in Figure 2. The Figure shows

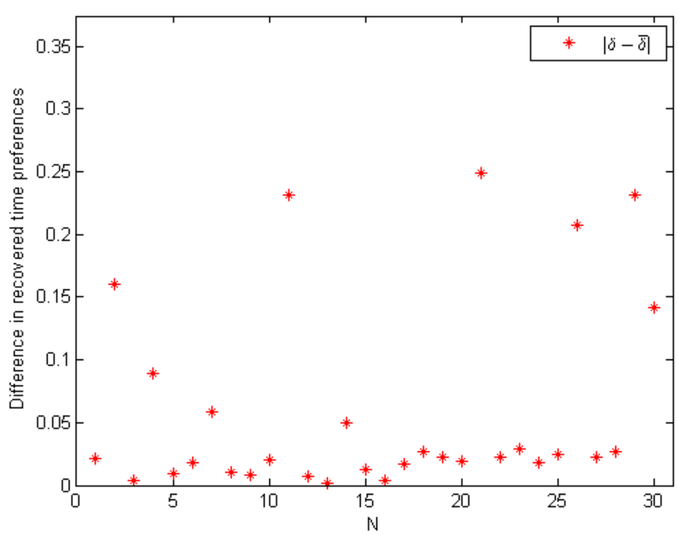


Figure 2: Mismatches in recovered time preferences.

that the time discount factors recovered in original specification and subjective specification generally differ, and often markedly differ from one another.[23]

We now compare the physical transition probabilities recovered on the same randomly generated $N = 30$ original $n = 3$-state systems and their respective consolidated $\overline{n} = 2$-state counterparts. The physical distributions are solved from eigen-problems (3) and (4), for $n \times n$ matrix A and $\overline{n} \times \overline{n}$ matrix $\overline{A}$. Given the particular consolidation scheme of this specific numerical example (see Figure 1), the most obvious pairwise transition probabilities to be compared are $Prob_{33}^{P}$ against $\overline{Prob}_{dd}^{P}$. We report the absolute value of their differences for $N = 30$ random trials in Figure 3. The Figure shows that even when state d is correctly

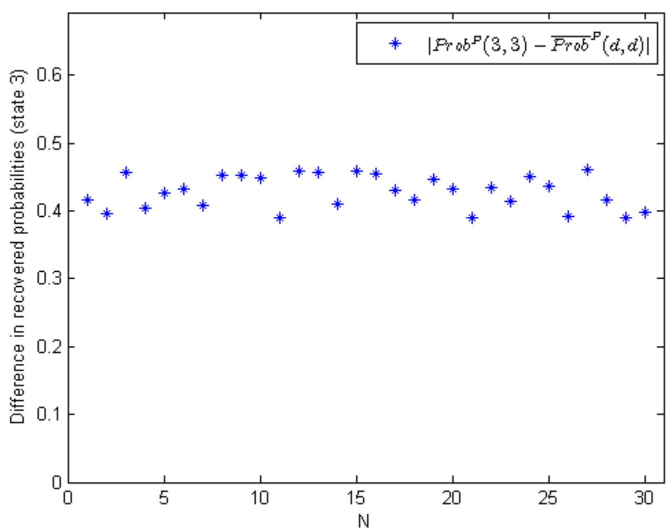


Figure 3: Mismatches in recovered beliefs on identically specified states of original and consolidated data.

specified in the subjectively specification $\overline{n} = 2$ (because d coincides with the pure state 3 in the original specification $n = 3$), the recovered transition probability $\overline{Prob}_{dd}^{P}$ is markedly off from the original counterpart $Prob_{33}^{P}$, which truly underlies the data.

[23]Their sizable difference is now plausible and is a consequence outcome of the arbitrages created in the consolidated system. In fact, the Perron-Frobernius theorem does not apply to matrix $\overline{A}$ having negative entries, and neither does the Recovery Theorem hold. Consequently, its largest eigenvalue $\overline{\delta}$ does not represent the time preference of the market.

Next, we compare the physical probabilities recovered on original and consolidated systems, for the transitions that are not identically specified in the two systems. In particular, referring to Figure 1, as the original states $\{1, 2\}$ are components of the single consolidated state u, we examine the following differences of the recovered transition probabilities,

$$\begin{aligned} \Delta P_1 &\equiv Prob^P(1,1) + Prob^P(1,2) - \overline{Prob}^P(u,u); \\ \Delta P_2 &\equiv Prob^P(2,1) + Prob^P(2,2) - \overline{Prob}^P(u,u). \end{aligned} \tag{16}$$

These differences are depicted in Figure 4 for $N = 30$ randomly generated price data sets mentioned above. The Figure shows that, for same trial, it appears often that the recovered

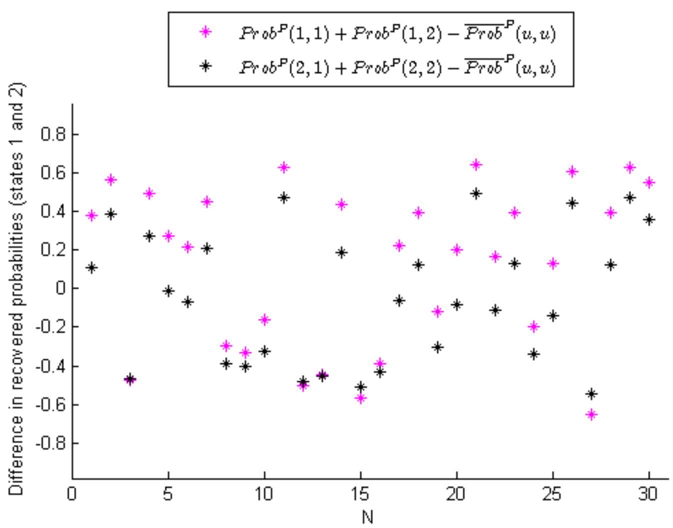


Figure 4: Mismatches in recovered beliefs on differently specified states of original and consolidated data.

probabilities in the consolidated specification $\overline{n} = 2$ are *uniformly* biased compared to those in the original specification $n = 3$, regardless of which original states 1 or 2 to be compared with the consolidated state u. That is, for majority of N trials, we see *both* biases defined in (16) have same sign; $\Delta P_1 \times \Delta P_2 > 0$. Together, the numerical results in Figures 3, 4 indicate that when there is a mismatch in the overall specifications n and $\overline{n}$, the recovered probabilities can differ for all types of transitions (either identically specified, or differently specified) in the two systems. These mismatches in the recovered statistics are not of numerical nature, as our formal results show next.

3.4 Specification issues: Formal results

Our formal analysis considers the consolidation of original n-state space into a $\overline{n}$-state space, where $n > \overline{n}$ (11). This consolidation is practical and indispensable routine in the recovery implementation because of both computing power limitation and unobserved "true" (i.e., data-generating) specification n of the market. The consolidation, and the subsequent recovery based on $\overline{n} \times \overline{n}$ consolidated AD price matrix $\overline{A}$ versus that based on $n \times n$ original AD price matrix A are prescribed in steps 0 through 3 of our thought experiment. The following Proposition establishes necessary and sufficient condition for the market's recovered characteristics (belief, time and risk preferences) on original n-state and consolidated $\overline{n}$-state specifications to be consistent with one another.

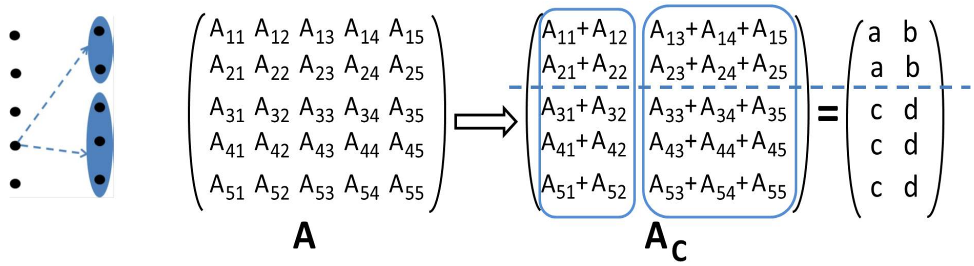
Proposition 1 (Consistent recovery) *Assume that the just-identified estimation procedure (7) is employed. The recoveries using $n \times n$ original AD price matrix A and using the $\overline{n} \times \overline{n}$ consolidated AD price matrix $\overline{A}$ are consistent if and only if, after the consolidation of columns of A (in the identical pattern of (12) which combines original n-state data set $\mathcal{A}_{1\uparrow}$ into $\overline{n}$–state data set $\overline{\mathcal{A}}_{1\uparrow}$), the resulting $n \times \overline{n}$ matrix A_c has identical rows within each consolidated block,*

$$\sum_{j \in \overline{j}} A_{ij} = \sum_{j \in \overline{j}} A_{i'j}, \quad \forall i, i' \in \overline{i}, \ \forall \overline{i}, \overline{j} \in \overline{\Omega}.$$

where i, j and $\overline{i}, \overline{j}$ denote original and consolidated states respectively (and $\overline{\Omega}$ denotes state partition after consolidation).

We first illustrate the thesis of this Proposition in the following specific scheme (Figure 3.4), leaving technical proof to Appendix A.1. Consider original $(n = 5)$–state price data set being consolidated into $(\overline{n} = 2)$–state set. The states' consolidation is as follows; $\{1, 2\} = \{u\}$,

$\{3,4,5\} = \{d\}$. In the data, this consolidation pattern is realized by summing columns 1, 2,



and 3, 4, 5 of the original price data set $\mathcal{A}_{1\uparrow}$ (6) to create respectively columns 1 and 2 of data set $\overline{\mathcal{A}}_{1\uparrow}$ (13). The identical consolidation pattern applied on 5×5 A means the summing of A's columns 1, 2, and 3, 4, 5 respectively, resulting in the 5×2 matrix A_c. Proposition 1 then states that the recoveries implemented on original and consolidated data are consistent if and only if A_c's first two rows (i.e., first consolidated block) are identical, and so are A_c's last three rows (i.e., last consolidated block).

The stated condition of the Proposition is intuitive and plausible. The necessary–and–sufficient condition simply requires that price $\sum_{j\in\bar{j}} A_{ij}$ of the AD-portfolio contract that starts from *any* original state i (within a consolidated state $\bar{i}$) and pays only in any states j within a consolidated state $\bar{j}$ be the same for all initial states $i \in \bar{i}$. Only under this condition the recovery implemented on consolidated data yields market characteristics (beliefs, time and risk preferences) that are consistent with those underlying the raw original price data. Clearly, the necessary–and–sufficient condition asserted by Proposition 1 is tight, hence the consistency of the recovery may hold only under very strict circumstance. This in turn implies that both unknown specification of discrete state space and the straightforward discretization of state space likely hinder the robustness of the recovery approach via first estimating full AD price matrices. The analytical result formalizes various exemplified numerical illustrations presented in previous section concerning the various consistency aspects of the recovery.

However, we note that Proposition 1 applies only for the just-identified estimation procedure (7), in which only $\overline{n}+1$ rows of consolidated data are employed in the recovery implementation. Consequently, it should come as little surprise that the information miss-out unknowingly committed by analysts choosing identification subjectively (studied in previous section) most likely[24] causes mismatches between recovered results based on different specifications. What if analysts instead employ all data in the original price data set $\mathcal{A}_{1\uparrow}$, while being unaware of the original specification n that underlies the data. The next proposition shows that even with infinite amount of perfect data, a best-fit recovery implemented on analysts' subjective specification may equally likely lead to spurious arbitrages and mismatches in recovered beliefs, time and risk preferences.

Proposition 2 (Least-squares recovery) *Assume that the OLS approach is employed. Let A be the $n \times n$ original AD price matrix and $\overline{A}$ be the $\overline{n} \times \overline{n}$ consolidated AD price matrix as usual. Then $\overline{A}$ satisfies the following formula:*

$$\overline{A} = \left(A_C^T A_C\right)^{-1} A_C^T A A_C,$$

where A_C is the consolidated version of the original matrix A defined in Proposition 1.

We first prove the proposition as follows. By the relation in (6), we obtain two equations below.

$$\begin{cases} \mathcal{A}_{2\uparrow} = \mathcal{A}_{1\uparrow} A \\ \mathcal{A}_{3\uparrow} = \mathcal{A}_{2\uparrow} A \end{cases} . \tag{17}$$

Performing the consolidation procedure shown in (13), the equations above become the following:

$$\begin{cases} \overline{\mathcal{A}}_{2\uparrow} = \overline{\mathcal{A}}_{1\uparrow} A_C \\ \overline{\mathcal{A}}_{3\uparrow} = \overline{\mathcal{A}}_{2\uparrow} A_C \end{cases} . \tag{18}$$

[24] That is, only when the condition of Proposition 1 holds, the recovery on consolidated date is consistent with that on original data.

Now applying the OLS approach (8) in the consolidated model[25], we get

$$\overline{A} = \left(\overline{\mathcal{A}}_{2\uparrow}^T \overline{\mathcal{A}}_{2\uparrow}\right)^{-1} \overline{\mathcal{A}}_{2\uparrow}^T \overline{\mathcal{A}}_{3\uparrow}^T. \tag{19}$$

Substituting (17) and (18) to the right hand side of (19) yields

$$\overline{A} = \left(A_C^T \overline{\mathcal{A}}_{1\uparrow}^T \overline{\mathcal{A}}_{1\uparrow} A_C\right)^{-1} A_C^T \overline{\mathcal{A}}_{1\uparrow}^T \overline{\mathcal{A}}_{1\uparrow} A A_C. \tag{20}$$

Let B be an $\overline{n} \times n$ matrix such that $AA_C A_C^T = A_C B$. To see the existence and uniqueness of B, pre-multiply the equation by A_C^T on both sides and we get

$$A_C^T A A_C A_C^T = A_C^T A_C B. \tag{21}$$

Since $n > \overline{n}$, we know that $A_C A_C^T$ is not invertible, but $A_C^T A_C$ is. Thus, taking inverse of $A_C^T A_C$ on both sides of (21) gives the expression for B as follows.

$$B = \left(A_C^T A_C\right)^{-1} A_C^T A A_C A_C^T. \tag{22}$$

We can see that B is uniquely determined, although it is singular. Now post-multiplying $A_C^T A_C$ to both sides of (20), we have

$$\overline{A} A_C^T A_C = \left(A_C^T \overline{\mathcal{A}}_{1\uparrow}^T \overline{\mathcal{A}}_{1\uparrow} A_C\right)^{-1} A_C^T \overline{\mathcal{A}}_{1\uparrow}^T \overline{\mathcal{A}}_{1\uparrow} A A_C A_C^T A_C \tag{23}$$

$$= \left(A_C^T \overline{\mathcal{A}}_{1\uparrow}^T \overline{\mathcal{A}}_{1\uparrow} A_C\right)^{-1} A_C^T \overline{\mathcal{A}}_{1\uparrow}^T \overline{\mathcal{A}}_{1\uparrow} A_C B A_C \tag{24}$$

$$= B A_C. \tag{25}$$

With equations (22) and (25) as well as the invertibility of $A_C^T A_C$, we conclude that $\overline{A} = \left(A_C^T A_C\right)^{-1} A_C^T A A_C$. This completes the proof.

[25] Equation (8) uses data from time $t = 1$. Since we have infinite amount of data and we assume the AD price matrix is time-invariant, it does not matter what the starting time is.

The formula in Proposition 2 provides a much more convenient way to obtain the consolidated AD price matrix $\overline{A}$ under the OLS approach. Traditionally, if we would like to use the best-fit approach on an infinite data set to get $\overline{A}$, we would have to add up certain columns of the original infinite data sets $\mathcal{A}_{1\uparrow}$ and $\mathcal{A}_{2\uparrow}$ via (12) to get the consolidated data sets $\overline{\mathcal{A}}_{1\uparrow}$ and $\overline{\mathcal{A}}_{2\uparrow}$ and then apply (8) to obtain $\overline{A}$. This traditional method is theoretically plausible, but is practically infeasible in two ways. First, the idea of infinite amount of perfect data cannot be implemented in reality because of the limitation of computing power. We could find a powerful machine to store enormous amount of data, but will never get to the point where we can reach the infinity. In addition, computing $\overline{\mathcal{A}}_{1\uparrow}^T\overline{\mathcal{A}}_{1\uparrow}$ would not be an easy task as $\overline{\mathcal{A}}_{1\uparrow}$ has a huge, if not infinite, number of rows. The same reasoning applies to computing $\overline{\mathcal{A}}_{1\uparrow}^T\overline{\mathcal{A}}_{2\uparrow}$. Second, more intuitively, deriving the consolidated AD price matrix using all of the available data seems to be a redundant task. Recall that under the thought experiment we start with the original AD price matrix A and use A to generate infinite amount of perfect data $\mathcal{A}_{1\uparrow}$; in other words, A can be viewed as if it was obtained from $\mathcal{A}_{1\uparrow}$ (and $\mathcal{A}_{2\uparrow}$) via (8). Therefore, the matrix A has incorporated all of the information contained in the price data sets. Notice also that the consolidated data sets are obtained from the original $\mathcal{A}_{1\uparrow}$ and $\mathcal{A}_{2\uparrow}$. It is then logical to find a way to derive $\overline{A}$ without repeating the same procedure as we did for A. Proposition 2 provides the answer to us, making the derivation more efficient and accurate.

Proposition 2 has several important consequences. First, the consolidated AD price matrix $\overline{A}$ may not have the same time preference as the original matrix A. (See Appendix A.2 for the proof that OLS will always give incorrect time preference for $n = 2$ reducing to $\overline{n} = 1$.) Moreover, in general the recovered physical probability distributions in these two case will differ. This shows that consolidation using OLS does not necessarily guarantee the consistency. Second, under this setting it is possible that the spurious arbitrage exists: some entry in $\overline{A}$ can be negative. To illustrate these defects, consider the following example. Suppose the original model has three states and we consolidate the first two states into u

and leave the third as d. Assume the original A is

$$A = \begin{pmatrix} 0.2820 & 0.0118 & 0.1034 \\ 0.2747 & 0.2306 & 0.3116 \\ 0.6037 & 0.0383 & 0.2939 \end{pmatrix}.$$

Using the formula in Proposition 2, we obtain the consolidated AD price matrix $\overline{A}$ below.

$$\overline{A} = \begin{pmatrix} 0.1246 & -0.0632 \\ 1.0621 & 0.7073 \end{pmatrix}$$

Furthermore, the time preference for A is 0.5791 and for $\overline{A}$ is 0.5494, and the differences in the recovered physical probability distributions are

$$\begin{aligned} &Prob^P(1,1) + Prob^P(1,2) - \overline{Prob}^P(u,u) = 0.3223 \\ &Prob^P(2,1) + Prob^P(2,2) - \overline{Prob}^P(u,u) = 0.3272 \\ &Prob^P(3,3) - \overline{Prob}^P(d,d) = 0.7800. \end{aligned}$$

This shows that the problems in consolidation using the just-identified approach persist and that introducing OLS does not essentially improve the situation; even with infinite and perfect data, incorrect specification of states can still lead us to inconsistent recovery and generate spurious arbitrage.

4 Continuous space-time setting

Deeper insights into the specification issues of the recovery implementation can be achieved by working with continuous state and time setting, in which ultimately markets operate. This investigation then also suggests proper procedures to discretize the state space to implement the recovery in consistent fashions. We start out to assume that market is driven by a stochastic state variable x_t in continuous time. For ease of exposition, we assume that x_t is one-dimensional and follows some (unknown ex-ante) diffusion process in either physical or risk-neutral measure,[26]

$$dx = x\mu_x dt + x\sigma_x dB_t = x\mu_x^Q dt + x\sigma_x dB_t^Q, \tag{26}$$

where B_t and B_t^Q are standard Brownian motion in physical and risk-neutral measure respectively. The drifts μ_x, μ_x^Q and volatility σ_x are to be recovered ex-post, and can be state-dependent (in form of processes adapted to natural filtration generated by B_t and B_t^Q). In special cases in which x is price of traded assets such as equities or equity indexes, then risk-neutral drift coincides with risk-free rate, $\mu_x^Q = r(x)$. All quantities in the market equilibrium are modeled as functions of this state variable, e.g. SDF $M_i = \frac{1}{z_i}$ (2) and interest rate r_i in the continuous state space setting reads $M(x_t) = \frac{1}{z(x_t)}$ and $r(x_t)$ respectively.

4.1 Infinitesimal operator and finite differencing

We recall that key to the recovery in discrete state space setting is the AD price matrix A. At the heart of the recovery in continuous state space setting is the infinitesimal operator $\mathcal{D}^Q$ associated with the risk-neutral state dynamic (26). When states and time are continuous, this operator plays the preeminent role of AD matrix in pricing all traded assets. The following Proposition establishes correspondence between these two objects in both short and long time horizons, generalizing an earlier result in Tran (2010).

Proposition 3 (Discrete and continuous state space mapping) *Let $\mathcal{D}^Q$ denote the the infinitesimal operator associated with state risk-neutral dynamic (26) which also includes risk-free discounting.*

$$\mathcal{D}^Q \equiv x\mu_x^Q \frac{d}{dx} + \frac{1}{2}x^2\sigma_x^2 \frac{d^2}{dx^2} - r(x) \tag{27}$$

[26]Higher dimensions and discontinuous dynamic (jumps) can also be incorporated.

Let A^t as always denote AD price matrix associated with any horizon t.[27] *The following correspondences relate the no-arbitrage pricing in discrete and continuous state space settings.*

$$A^{dt} \longleftrightarrow \mathbb{1} + dt\mathcal{D}^Q; \qquad\qquad A^t \longleftrightarrow e^{\int^t ds\mathcal{D}^Q}. \tag{28}$$

The exponential operator $e^{\int^t ds\mathcal{D}^Q}$ should be understood as the power series (51) of $\mathcal{D}^Q$. Technical details and the proof of this Proposition are given in Appendix B. By resorting to the underlying continuous dynamic, the mapping (28) gives us a way to visualize and model the full AD price matrix for any time horizons, even for those that are shorter than what available in actual data. Furthermore, large and discontinuous changes in state variable x can also be conveniently modeled by incorporating jump dynamic into the infinitesimal operator $\mathcal{D}^Q$.

An immediate application of Proposition 3 is to explore the Recovery Theorem in continuous state space setting. To this end, assume that $(\delta, \{z_i\}_{i=1}^n)$ is eigenpair solution of the fundamental equation (3), $Az = \delta z$, in discrete state space. When we apply the correspondence in (28) to this eigenstate z for both time horizons,

Short-horizon						Long-horizon		
$A^{dt}z$	$=$	$\delta^{dt}z$	$\approx$	$(\mathbb{1} + \log\delta dt)\, z$		$A^t z$	$=$	$\delta^t z$
		$\updownarrow$					$\updownarrow$	
$e^{dt\mathcal{D}^Q}z$	$\approx$	$\left(\mathbb{1} + dt\mathcal{D}^Q\right) z$	$=$	$(1 - \rho dt)\, z$		$e^{\int^t ds\mathcal{D}^Q}z$	$=$	$e^{-\rho t}z$

From these follows the fundamental recovery equation in continuous state space,

$$\mathcal{D}^Q z(x) = -\rho z(x), \qquad \text{where} \qquad \rho \equiv -\log\delta \Longleftrightarrow \delta \equiv e^{-\rho}$$

or more explicitly

$$x\mu_x^Q \frac{dz(x)}{dx} + \frac{1}{2}x^2\sigma_x^2 \frac{d^2 z(x)}{dx^2} - r(x)z(x) = -\rho z(x). \tag{29}$$

[27] Hence A_{ij}^{dt} is the current price at state i and time t of the AD security that pays \$1 only if state at $t+dt$ is j.

It is reassuring that δ and ρ are standard conventions for time discount factor in discrete and continuous time setting respectively. Tran (2010) shows that all stochastic discount factors $M(x)$ (2) being functions of state variable, of which the Recovery Theorem's setting is a particular case, must satisfy the differential equation (29). Carr and Yu (2012) show that this differential equation (29) has *unique* positive solution when (i) state variable X is a bounded diffusion process, and (ii) appropriate Sturm-Liouville boundary conditions are imposed on the boundary of the support of X, thus extending the Recovery Theorem to continuous time and state setting. One may also obtain the uniqueness of the recovery for compact linear operator using Krein-Rutman theorem (appendix C.2), a direct generalization of Perron-Frobenius theorem to continuous state space setting.

The recovery in the continuous state space setting, however, requires knowledge of risk-neutral state dynamic $\{\mu_x^Q, \sigma_x\}$ and short rate process $r(x)$.[28] Literature preexisting the advent of the Recovery Theorem has offered several procedures to estimate these statistics. When there is a market trading options contingent on state variables x's,[29] Bakshi, Kapadia and Madan (2003) (see also appendix D) derive model-independent formulas for first four risk-neutral moments of x in term of integrals of option prices. More generally, when x's (and short rate r) are observable and market is complete so that contracts of payoffs contingent on x at maturity are traded, a *cross section* of prices (and risk-neutral pricing) of these contracts can identify risk-neutral state dynamic $\{\mu_x^Q, \sigma_x\}$.[30] When x's are latent state variables, dynamic term structure literature jointly models risk-neutral state dynamic and short rate in some reduced forms and estimates these processes efficiently using maximum likelihood methods on fixed-income price data (see e.g. Singleton, 2006).

We now embark on an even more explicit map between A and $\mathcal{D}^Q$, which also sheds light

[28] In the special case in which short rate is state-independent; $r_x = r$, the unique positive eigenstate solution of 29 is a constant function $z(x) = z$, corresponding to the eigenvalue $\rho = r$. That is, recovered market's discounting is risk-neutral if short rate is state-independent. The same result is obtained by Ross (2013) in discrete state space setting.

[29] When state variable x's are the prices of traded assets, no-arbitrage implies that $\mu_x^Q = r(x)$.

[30] Dubynskiy and Goldstein (2013) specifically show that price $P(dx)$ of contract of payoff dx identifies μ_x^Q: $P(dx) = E_t^Q\left[e^{-r_t dt}dx\right] = x\mu_x^Q dt$; price $P(dx^2)$ of contract of payoff $(dx)^2$ identifies σ_x^2: $P(dx^2) = E_t^Q\left[e^{-r_t dt}dx^2\right] = x^2\sigma_x^2 dt$, and so on.

into the causes of specification issues of discrete state space studied in the previous section. Specifically, we consider the finite difference representation of infinitesimal operator (27),

$$\begin{aligned} \mathcal{D}^Q z(x) = & - r(x)z(x) + \mu_x^Q x \frac{z(x+dx) - z(x-dx)}{2dx} \\ & + \frac{1}{2}\sigma_x^2 x^2 \frac{z(x+dx) - 2z(x) + z(x-dx)}{(dx)^2}. \end{aligned} \tag{30}$$

This representation, together with the correspondence $A^{dt} \leftrightarrow \left(\mathbb{1} + dt\mathcal{D}^Q\right)$, gives rise to the following explicit matrix form for the AD matrix associated with horizon dt,

$$A^{dt} = \begin{pmatrix} X & Y & 0 & 0 & \dots & 0 & 0 & 0 \\ Z & X & Y & 0 & \dots & 0 & 0 & 0 \\ 0 & Z & X & Y & \dots & 0 & 0 & 0 \\ \vdots & & & \ddots & & & & \vdots \\ \vdots & & & & \ddots & & & \vdots \\ 0 & 0 & 0 & \dots & Z & X & Y & 0 \\ 0 & 0 & 0 & \dots & 0 & Z & X & Y \\ 0 & 0 & 0 & \dots & 0 & 0 & Z & X \end{pmatrix} \quad \text{where} \quad \begin{cases} X \equiv 1 - r(x)dt - \sigma_x^2 x^2 \frac{dt}{(dx)^2} \\ \\ Y \equiv \frac{1}{2}\sigma_x^2 x^2 \frac{dt}{(dx)^2} + \frac{1}{2}\mu_x^Q x \frac{dt}{dx} \\ \\ Z \equiv \frac{1}{2}\sigma_x^2 x^2 \frac{dt}{(dx)^2} - \frac{1}{2}\mu_x^Q x \frac{dt}{dx} \end{cases} \tag{31}$$

Note that X, Y, Z generally may vary with x (so long as are $r(x)$, μ_x^Q, σ_x), and thus with their position in the AD price matrix.

4.2 Recovery via risk-neutral dynamic and finite differencing

To exemplify the possible causes of specification issues in discrete state setting, in what follows we first assume that the market is driven by an objective diffusion state variable (26) of risk-neutral dynamic $\{\mu_x^Q, \sigma_x\}$ in continuous time and state setting. For ease of exposition, we also assume that μ_x^Q, σ_x are constants (i.e., state independent), though our analysis applies to more general setting of state-dependent dynamics.

Arbitrages and causes

The finite differencing representation (31) of AD price matrix A^{dt} with time horizon dt clearly shows that time and state partitioning (i.e., consolidating) in approximate discrete version of the market can give rise to negative AD prices, and thus arbitrages. Indeed, X, Y, Z in (31) are positive only when $\frac{dx}{x}$ and dt jointly satisfy,

$$\left|\frac{\sigma_x^2}{\mu_x^Q}\right| > \frac{dx}{x} > \sigma_x\sqrt{\frac{dt}{1-r(x)dt}}. \tag{32}$$

Analysts unaware of dynamic $\{\mu_x^Q, \sigma_x\}$ may unknowingly choose $\{dx, dt\}$ such that one (both) of the above inequalities is (are) violated. As a result, arbitrages may spuriously arise though they are absent in the market at the fundamental level of continuous time and states. However, we note that risk-neutral dynamic $\{\mu_x^Q, \sigma_x\}$ can be reliably estimated for various asset markets via various estimation procedures. Our current investigation then suggests to first estimate the risk-neutral dynamic and then, based on this estimates, to consolidate states and time in a harmonious way to avoid creating arbitrages outright. In particular, highly stable state variable (σ_x^2 small) may require fine state partition ($\frac{dx}{x}$ small). We will revisit the recovery via risk-neutral dynamic estimation below.

Mimicking trinomial process

The finite differencing representation of AD price matrix (31) is a strong reminder of the trinomial approximation of the underlying diffusion process, all in risk-neutral measure. In this mimicking framework, from state x at t, one of three states $\{x - dx, x, x + dx\}$ can be reached at time $t+dt$ with *risk-neutral* probabilities $\{p_d^Q, p_m^Q, p_u^Q\}$ respectively,[31]

$$\begin{cases} p_d^Q = \frac{Z}{X+Y+Z} = \frac{Z}{1-r(x)dt} \\ p_m^Q = \frac{X}{X+Y+Z} = \frac{X}{1-r(x)dt} \\ p_u^Q = \frac{Y}{X+Y+Z} = \frac{Y}{1-r(x)dt} \end{cases} \Longrightarrow \begin{cases} E_t^Q[\frac{dx}{x}] = \mu_x^Q dt \\ \\ E_t^Q[(\frac{dx}{x})^2] = \sigma_x^2 dt \end{cases} \tag{33}$$

where X, Y, Z are given in (31). It is reassuring that these probabilities correctly reproduce the the first two moments of the stochastic growth for state variables. Furthermore, the same conditions (32) enforcing no-arbitrages also assure positivity for all feasible transition probabilities. Hence, the representation (31) describes indeed a proper trinomial process mimicking the "true" market continuous dynamic (26) in risk-neutral measure. The consistent convergence of *any* (discrete) trinomial process to the underlying (continuous) diffusion process in proper limits[32] of $dx \to 0$, $dt \to 0$ is important to our approach to recovery via Q-dynamic estimation and finite differences.

Recovering physical dynamic

To the aim of uniquely recovering physical dynamic $\{\mu_x^P, \sigma_x\}$ (26) of the market, following Ross (2013) and Carr and Yu (2012) we assume ad-hoc bounded support $\{\underline{x}, \overline{x}\}$ for state variable and apply boundary conditions of Sturm-Liouville's type.[33] In continuous and finite-difference settings, these conditions respectively read,

$$\begin{cases} \kappa z(\underline{x}) - \frac{dz}{dx}\Big|_{\underline{x}} = 0 \\ \kappa z(\overline{x}) - \frac{dz}{dx}\Big|_{\overline{x}} = 0 \end{cases} \longleftrightarrow \begin{cases} \kappa z_0 - \frac{z_1 - z_0}{dx} = 0 \\ \kappa z_n - \frac{z_n - z_{n-1}}{dx} = 0 \end{cases} \tag{34}$$

[31] Recall that the risk-neutral one-period transition probabilities are ratios of t-period AD prices: $prob^Q(i,0;j,t) = \frac{A_{ij}^t}{\sum_j A_{ij}^t}$. For short horizon limit, $(t = dt)$, we furthermore have $\sum_j A_{ij}^{dt} = e^{-r_i dt}$.

[32] That is, in the limiting process, dx and dt need to satisfy the restrictions (32) at all time.

[33] Another possibility, alternative to imposing boundary conditions, is to invoke Krein-Rutman theorem (appendix C.2) and submit to its underlying assumptions.

Technically, in finite differencing representation, these boundary conditions amount equivalently to fixing the upper-left-most and lower-right-most entries of AD price matrix (31). Conceptually, these boundary conditions can be interpreted as to fixing the market's *absolute risk aversion* to κ when state variable reach $\underline{x}$ or $\overline{x}$. Indeed, in representative agent's framework and when state variable is aggregate consumption, SDF M is marginal utility U'. The relation (2) then translates the above boundary conditions into a restriction on market's absolute risk aversions at the boundaries,

$$\frac{-U''(\underline{x})}{U'(\underline{x})} = \frac{-U''(\overline{x})}{U'(\overline{x})} = \kappa.$$

Anywhere else inside the support $(\underline{x}, \overline{x})$, risk aversion does vary in a pattern to be pinned down endogenously by the recovery process.

The boundary conditions (34) give rise to a *unique* positive solution to the eigen-problem (28) of the Recovery Theorem,

$$A^{dt} z = (\mathbb{1} + dt\mathcal{D}^Q)z = \delta^{dt} z \tag{35}$$

In the finite difference approach, this fundamental equation has explicit matrix form (31) once $Q-$dynamic $\{\mu_x^Q, \sigma_x\}$ have been estimated.[34] This eludes the step of consolidating states and backing out one-period AD price matrix indirectly from available derivative prices contracted on today's state i over different maturities, which gives rise to various specification issues discussed in previous section. This is an important advantage of finite difference procedure to the recovery. Computationally, solving differential equation (35) using finite-difference method is highly efficient once the state and time partitions satisfy convergence conditions (32). For the sake of an illustration, we consider a numerical thought experiment, in which we know the risk-neutral dynamic μ_x^Q, σ_x and $r(x)$ precisely. Figure 5 plots two numerical solutions for eigenstates $z(x)$ of A^{dt} associated with its largest eigenvalue δ^{dt}. The solutions

[34] Recall that we do also need to know or assume short rate process $r(x)$ in continuous state space setting, because $r(x)$ shows up in both expression (27) for $\mathcal{D}^Q$ and (31) for A^{dt}.

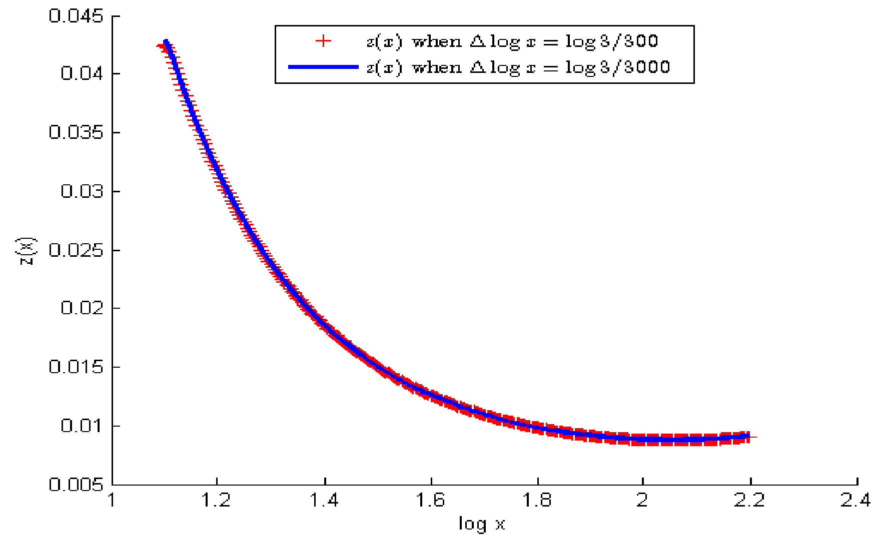


Figure 5: Numerical solutions $z(x)$ of eigen-problem $A^{dt}z = \delta^{dt}z$ (35) for two finite-difference schemes, respectively with 300 and 3000 spacing steps. State space support $(\underline{x}, \overline{x}) = (3, 9)$, Q-dynamic $\mu_x^Q = 1.24\%$, $\sigma_x = 1.14\%$, time step $dt = 10^{-3}$, affine short rate $r_x = a + b \times x = -0.0448 + 0.0431 \times x$. The boundary conditions (34) with $\kappa = 0.5$ are realized in finite-difference setting by choosing left-most entries of first row, and right-most entries of last row of matrix (31) such that its first row sums to $1 - r_{\underline{x}}dt$, its last row sums to $1 - r_{\overline{x}}dt$. A similar issue of endogenous boundary conditions in the recovery implementation (and in general equilibrium setting) is discussed first in Dubynskiy and Goldstein (2013).

correspond to partitioning the same support $(\underline{x}, \overline{x}) = (3, 9)$ of state variable x into $\overline{n} = 300$ and $n = 3000$ intervals respectively,[35] while keeping unchanged $dt = 10^{-3}$ for time step and $\kappa = 0.5$ for boundary conditions (34). Short rate $r(x)$ is state-dependent (affine) to assume away from recovering the very special and unrealistic risk-neutral discounting for the market (see also footnote 28). Figure 5 shows that the numerical solutions for $z(x)$ converge well despite of being generated from very different spacing steps dx. The time discount rates recovered from two schemes also agree well; $\delta_{300} = \delta_{3000} = 0.9997$. Note that the current convergence happens when state space partition gets finer on a *fixed* support, while conditions (32) are enforced at all time. Hence it is different from an interesting

[35] We work with $\log x$, and thus the spacing steps are $d \log x = \frac{\log \overline{x} - \log \underline{x}}{300} = \frac{\log 3}{300}$ and $d \log x = \frac{\log 3}{3000}$ respectively.

counter-example of Dubynskiy and Goldstein (2013), who demonstrate the coexistence of convergence and econometric fragility by *extending* state space support.[36] Then follow from eigenstate solution $z(x)$ and risk-neutral dynamic $\{\mu_x^Q, \sigma_x\}$ the market price of risk η_x,

$$\eta_x^2 = \frac{1}{dt} E_t^P \left[\left(\frac{dM}{M} \right)^2 \right] = \frac{1}{dt} E_t^P \left[\left(\frac{dz}{z} \right)^2 \right] = \sigma_x^2 \left(\frac{1}{z} \frac{dz}{dx} \right)^2 \Longrightarrow \eta_x = \sigma_x \frac{1}{z} \frac{dz}{dx}.$$

and physical dynamic $\{\mu_x^P, \sigma_x\}$

$$\mu_x^P = \mu_x^Q + \sigma_x \eta_x = \mu_x^Q + \sigma_x^2 \frac{1}{z} \frac{dz}{dx} = \mu_x^Q + \sigma_x^2 \frac{1}{z(x)} \frac{z(x+dx) - z(x-dx)}{2dx}.$$

where the last expression is a finite-difference approximation. Finally, note that the diffusion invariance principle simply implies that volatility σ_x is the same for both risk-neutral and physical measure.

Recovering physical distribution (i.e., subjective belief)

After obtaining eigenstate $z(x)$ associated with the largest eigenvalue of AD price matrix A^{dt} in finite difference approximation, there are at least two approaches to recovering physical distribution $Prob^P(x)$. The first approach, which is true to the discretized nature of the finite differencing, employs the short-horizon version of relation (4) to build the short-horizon physical transition probability matrix

$$Prob^{P,dt}(i,j) = \frac{A_{ij}^{dt} z_j}{\delta^{dt} z_i},$$

where A^{dt} is (31), and its eigenpair $\{\delta^{dt}, z(x)\}$ has been readily numerically computed in previous step. The physical transition probability for longer horizon obtains by raising the

[36] Also key to their counter-example is particular boundary conditions that make *all* rows of AD price matrix have identical sum.

short-horizon transition probability matrix $[Prob^{P,dt}]$ to appropriate power of $\frac{t}{dt}$,

$$Prob^{P,t}(i,j) = \left[\left(Prob^{P,dt}\right)^{\frac{t}{dt}}\right]_{i,j} = \frac{A^t_{ij} z_j}{\delta^t z_i}.$$

The second approach (Dubynskiy and Goldstein, 2013, and Tran, 2010), which is true to the continuous nature of the market, employs the Kolmogorov backward equation to obtain the physical transition probability *density* $\pi(x,t;y,T)$,

$$\left(\mathcal{D}^P + \frac{\partial}{\partial t}\right)\pi(x,t;y,T) \equiv \left(\mu^P_x x \frac{\partial}{\partial x} + \frac{1}{2}\sigma^2_x x^2 \frac{\partial^2}{\partial x^2} + \frac{\partial}{\partial t}\right)\pi(x,t;y,T) = 0,$$

or in finite difference representation,

$$\frac{1}{2}\sigma^2_x x^2 \frac{\pi(x+dx,t;y,T) - 2\pi(x,t;y,T) + \pi(x-dx,t;y,T)}{(dx)^2}$$

$$+\mu^P_x x \frac{\pi(x+dx,t;y,T) - \pi(x-dx,t;y,T)}{2dx} + \frac{\pi(x,t+dt;y,T) - \pi(x,t-dt;y,T)}{2dt} = 0.$$

where physical dynamic $\{\mu^P_x, \sigma_x\}$ has been readily numerically computed in previous step. A nice feature of this second approach to physical distribution is that the Kolmogorov backward differential equation directly addresses and accommodates the variability of initial state x (while fixing final state y). The same variability has proved elusive vis-à-vis price data, which at any point in time is available only for financial assets contracted on a single current initial state x. Though, the computational cost of this approach is the need to solve for an extra finite difference equation involving transition probability density $\pi(x,t;y,T)$ above.

In summary, the Recovery Theorem can be bonded to the fundamental *dynamic* of state variables in physical measure and in continuous setting of the market, subject to appropriate recovery assumptions and boundary conditions. In this approach, first the risk-neutral state dynamic $\{\mu^Q_x, \sigma_x\}$ and short rate process $r(x)$ need be reliably determined (e.g., by empirical estimation on modeling or reduced-form assumptions). Then the finite difference representation of the key recovery eigen-problem need be solved numerically. The slicing of time and slicing of state space need to be harmonious (i) with one another in order not to give rise to spurious arbitrages, and (ii) with the numerical precision of the estimated $\{\mu_x^Q, \sigma_x, r(x)\}$. Finally, physical state dynamic $\{\mu_x^P, \sigma_x\}$, price of risk η_x, physical transition probability at short and long horizon need be computed from solution of the unique positive eigenstate $z(x)$ and the risk-neutral state dynamic.

5 Conclusion

The most natural procedure to implement the Recovery Theorem involves solving for an effective one-period AD price matrix from current prices of available financial assets maturing at different future dates. This step requires a specification of effective AD price matrix's the dimension, which amounts equivalently to partitioning and consolidating state space into several (composite) states. We show that different specifications may lead to permanent loss of information in price data, to arbitrages in the pricing of different effective AD matrices, and to discrepancies in subsequent market's characteristics recovered on different AD matrices.

We propose first to estimate the risk-neutral dynamic of state variable, using one of the procedures of preexisting literature. This estimated risk-neutral distribution then gives guidance to sampling and finite differencing time and state space in harmonious fashion not to create arbitrages spuriously. Market price of risk, physical state dynamic, and physical transition probability density then can be obtained by solving the finite difference representation of the fundamental eigen-problem of the Recovery Theorem.

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Appendices

A Proofs

A.1 Proof of Proposition 1

A.1.1 Sufficiency for any n reducing to $\overline{n} < n$

Denote A as the n-state AD price matrix and $\overline{A}$ as the $\overline{n}$-state one. Define $\mathcal{J} \equiv \{\mathcal{J}_k\}_{k=1}^{\overline{n}}$ to be the set containing all consolidated states, where each $\mathcal{J}_k$ is one consolidated state consisting of original states. We require that $\mathcal{J}_k$ be "increasing" in k, meaning that $\max(\mathcal{J}_{k_1}) < \min(\mathcal{J}_{k_2})$ if $k_1 < k_2$. For example, if $n = 5$ and we group the first two states together and last three states together; then $\mathcal{J}_1 = \{1, 2\}$ and $\mathcal{J}_2 = \{3, 4, 5\}$. The sufficient condition for consistent recovery states that for any $\mathcal{J}_{k_1}, \mathcal{J}_{k_2} \in \mathcal{J}$, with arbitrary $j_1, j_2 \in \mathcal{J}_{k_1}$, the matrix A satisfies $\sum_{j \in \mathcal{J}_{k_2}} A_{j_1,j} = \sum_{j \in \mathcal{J}_{k_2}} A_{j_2,j}$.

Lemma 1 *Assume neither of the price data sets $\mathcal{A}_{1\uparrow}$ and $\overline{\mathcal{A}}_{1\uparrow}$ under the just-identified approach is singular, and A fulfills the sufficient condition above. The AD price matrix $\overline{A} = (\overline{A}_{p,q})$ after consolidation satisfies $\overline{A}_{p,q} = \sum_{j \in \mathcal{J}_q} A_{m,j}$ for any $m \in \mathcal{J}_p$.*

Proof of Lemma 1. Let the current state be i. According to (12), we have $\sum_{j \in \mathcal{J}_k} A^t_{i,j} = \overline{A}^t_{i,k}$ for any t and any $\mathcal{J}_k \in \mathcal{J}$. Since $\mathcal{A}_{2\uparrow} = \mathcal{A}_{1\uparrow} A$, for any $j \in \mathcal{J}_k$ we have

$$A^t_{i,j} = \sum_{m=1}^{n} A^{t-1}_{i,m} A_{m,j}. \tag{36}$$

Summing up both sides of (36) over all $j \in \mathcal{J}_k$ and using the sufficient condition, we obtain the following:

$$\begin{aligned} \overline{A}^t_{i,k} \equiv \sum_{j \in \mathcal{J}_k} A^t_{i,j} = \sum_{j \in \mathcal{J}_k} \sum_{m=1}^{n} A^{t-1}_{i,m} A_{m,j} = \sum_{m=1}^{n} \sum_{j \in \mathcal{J}_k} A^{t-1}_{i,m} A_{m,j} \\ = \sum_{m=1}^{n} \left(A^{t-1}_{i,m} \sum_{j \in \mathcal{J}_k} A_{m,j} \right) = \sum_{h=1}^{\overline{n}} \left(\sum_{m \in \mathcal{J}_h} A^{t-1}_{i,m} \sum_{j \in \mathcal{J}_k} A_{m,j} \right). \end{aligned} \tag{37}$$

Now for any $\mathcal{J}_k, \mathcal{J}_h \in \mathcal{J}$ and any $m \in \mathcal{J}_h$, define an $\overline{n} \times \overline{n}$ matrix $\overline{A}$ whose hkth entry is $\overline{A}_{h,k} \equiv \sum_{j \in \mathcal{J}_k} A_{m,j}$. This matrix is well-defined because of the sufficient condition stated above. Thus, equation (37) implies $\overline{A}^t_{i,k} = \sum_{j=1}^{\overline{n}} \overline{A}^{t-1}_{i,j} \overline{A}_{j,k}$ for any t and $\mathcal{J}_k \in \mathcal{J}$, meaning that $\overline{\mathcal{A}}_{2\uparrow} = \overline{\mathcal{A}}_{1\uparrow}\overline{A}$. As $\overline{\mathcal{A}}_{1\uparrow}$ is non-singular, we know that $\overline{A}$ is unique and it must be the consolidated AD price matrix. ■

Lemma 2 *Assume the original AD price matrix A satisfies the sufficient condition. The consolidated AD price matrix $\overline{A}$ can be obtained from Lemma 1. Let the time preference for $\overline{A}$ be $\overline{\delta}$ and its associated eigenstate be $\overline{z} = [\overline{z}_1, \cdots, \overline{z}_{\overline{n}}]'$. Define $N_k =| \mathcal{J}_k |$ for $k = 1, \cdots, n$. Then, $\overline{\delta}$ is also the time preference for A. Moreover, its associated eigenstate z is*

$$z = [\underbrace{\overline{z}_1, \cdots, \overline{z}_1}_{N_1}, \underbrace{\overline{z}_2, \cdots, \overline{z}_2}_{N_2}, \cdots, \underbrace{\overline{z}_{\overline{n}-1}, \cdots, \overline{z}_{\overline{n}-1}}_{N_{\overline{n}-1}}, \underbrace{\overline{z}_{\overline{n}}, \cdots, \overline{z}_{\overline{n}}}_{N_{\overline{n}}}].$$

Proof of Lemma 2. Pick an arbitrary row k in $\overline{A}$. Since $\overline{\delta}$ is an eigenvalue with eigenstate $\overline{z}$, we have

$$\overline{A}_{k,1}\overline{z}_1 + \overline{A}_{k,2}\overline{z}_2 + \cdots + \overline{A}_{k,\overline{n}-1}\overline{z}_{\overline{n}-1} + \overline{A}_{k,\overline{n}}\overline{z}_{\overline{n}} = \overline{\delta}\overline{z}_k. \tag{38}$$

By Lemma 1, we can rewrite (38) as

$$\forall m \in \mathcal{J}_k, \qquad \sum_{j\in\mathcal{J}_1} A_{m,j}\overline{z}_1 + \sum_{j\in\mathcal{J}_2} A_{m,j}\overline{z}_2 + \cdots + \sum_{j\in\mathcal{J}_{\overline{n}-1}} A_{m,j}\overline{z}_{\overline{n}-1} + \sum_{j\in\mathcal{J}_{\overline{n}}} A_{m,j}\overline{z}_{\overline{n}} = \overline{\delta}\overline{z}_k. \tag{39}$$

Since k was chosen arbitrarily, we know that (39) holds for all $k = 1, \cdots, \overline{n}$. Hence, $\overline{\delta}$ is the time preference for A and z, defined in the statement, is its associated eigenstate. ■

According to Lemma 2, we have shown the sufficiency implies identical time preference for both models. To establish consistency in the recovery, we will prove that the two models have the same probability distribution.

Consider two arbitrary states, k_1 and k_2, in the consolidated model. The physical probability from k_1 to k_2 is defined as follows.

$$\overline{Prob}^P(k_1, k_2) = \frac{\overline{A}_{k_1,k_2}\overline{z}_{k_2}}{\overline{\delta}\overline{z}_{k_1}}. \tag{40}$$

Choose any original state $j_1 \in \mathcal{J}_{k_1}$. Consider $\sum_{j_2 \in \mathcal{J}_{k_2}} Prob^P(j_1, j_2)$. By definition,

$$\sum_{j_2 \in \mathcal{J}_{k_2}} Prob^P(j_1, j_2) = \frac{\sum_{j_2 \in \mathcal{J}_{k_2}} A_{j_1, j_2} z_{j_2}}{\bar{\delta} z_{j_1}}. \tag{41}$$

By Lemma 2, $z_{j_1} = \bar{z}_{k_1}$ and $z_{j_2} = \bar{z}_{k_2}$ $\forall j_2 \in \mathcal{J}_{k_2}$; then by Lemma 1, $\sum_{j_2 \in \mathcal{J}_{k_2}} A_{j_1, j_2} = \overline{A}_{k_1, k_2}$. Thus, equation (41) becomes

$$\sum_{j_2 \in \mathcal{J}_{k_2}} Prob^P(j_1, j_2) = \frac{\overline{A}_{k_1, k_2} \bar{z}_{k_2}}{\bar{\delta} \bar{z}_{k_1}} = \overline{Prob}^P(k_1, k_2). \tag{42}$$

Because (42) holds for any k_1 and k_2, we conclude that the two models have the identical physical probability distribution. Along with the identical time preference $\bar{\delta}$, we have proved the consistency in the recovery. ■

A.1.2 Necessity for $n = 2$ reducing to $\overline{n} = 1$

Let the original AD price matrix be $A = \begin{pmatrix} A_{1,1} & A_{1,2} \\ A_{2,1} & A_{2,2} \end{pmatrix}$. Without loos of generality, we set the current state $i = 1$ and then generate a 2×2 price data sets $\mathcal{A}_{1\uparrow}$ and $\mathcal{A}_{2\uparrow}$ under the just-identified approach:

$$\mathcal{A}_{1\uparrow} = \begin{pmatrix} A_{1,1} & A_{1,2} \\ (A_{1,1})^2 + A_{1,2}A_{2,1} & (A_{1,1} + A_{2,2})A_{1,2} \end{pmatrix},$$

$$\mathcal{A}_{2\uparrow} = \begin{pmatrix} (A_{1,1})^2 + A_{1,2}A_{2,1} & (A_{1,1} + A_{2,2})A_{1,2} \\ [(A_{1,1})^2 + A_{1,2}A_{2,1}]A_{1,1} & [(A_{1,1})^2 + A_{1,2}A_{2,1}]A_{1,2} \\ +(A_{1,1} + A_{2,2})A_{1,2}A_{2,1} & +(A_{1,1} + A_{2,2})A_{1,2}A_{2,2} \end{pmatrix}$$

The largest eigenvalue of A can be computed in the closed form as

$$\delta = \frac{A_{1,1} + A_{2,2} + \sqrt{(A_{1,1} - A_{2,2})^2 + 4A_{1,2}A_{2,1}}}{2}.$$

After consolidating two states into one we obtain

$$\overline{\mathcal{A}}_{2\uparrow} = \overline{\mathcal{A}}_{1\uparrow} \begin{pmatrix} x \\ y \end{pmatrix},$$

where $\overline{\mathcal{A}}_{1\uparrow}$ and $\overline{\mathcal{A}}_{2\uparrow}$ are 2×1 consolidated price data sets, i.e.,

$$x = \frac{\overline{A}_{i,1}^2}{\overline{A}_{i,1}} = \frac{A_{i,1}^2 + A_{i,2}^2}{A_{i,1} + A_{i,2}} \qquad \text{and} \qquad y = \frac{\overline{A}_{i,1}^3}{\overline{A}_{i,1}^2} = \frac{A_{i,1}^3 + A_{i,2}^3}{A_{i,1}^2 + A_{i,2}^2}.$$

Since we have assumed the recovery in the reduced system is consistent, we must have correct discount factors and subjective probabilities. As the probabilities in the one-state world is always equal to one, we are left with the discount factor only, which yields that $\overline{\delta} \equiv x = y = \delta$. Setting $\delta = x$, we get $4A_{1,2}[(A_{1,1} + A_{1,2}) - (A_{2,1} + A_{2,2})](A_{1,1}A_{2,2} - A_{1,2}A_{2,1}) = 0$; setting $\delta = y$, we get $4A_{1,2}[(A_{1,1} + A_{1,2}) - (A_{2,1} + A_{2,2})](A_{1,1}A_{2,2} - A_{1,2}A_{2,1})^2 = 0$. In order to have these two equations to hold simultaneously, we must require either $A_{1,1} + A_{1,2} = A_{2,1} + A_{2,2}$ or $A_{1,1}A_{2,2} - A_{1,2}A_{2,1} = 0$. The latter case incurs the singularity of the original *AD* price matrix, which is a contradiction to the assumption of unique recovery. Hence, the necessary condition in this example is that the sums of the two rows are identical. ■

A.1.3 Necessity for $n = 3$ reducing to $\overline{n} = 2$, and beyond

Suppose the three-state system can be consistently reduced to a two-state one by consolidating states 1 and 2 into state u and leaving state 3 as state d via (12). Denote the *AD* price matrices as A and $\overline{A}$ respectively. As the consolidation is consistent, define δ^* to be the common discount factor, i.e., the largest eigenvalue for both cases. Let the eigenstate associated to δ^* be z for A and $\overline{z}$ for $\overline{A}$ as usual. The definition of the physical distribution in (4) gives the following equations.

$$\begin{cases} Prob^P(1,1) + Prob^P(1,2) = \dfrac{1}{\delta^*}\dfrac{A_{1,1}z_1 + A_{1,2}z_2}{z_1} \\ \overline{Prob}^P(u,u) = \dfrac{1}{\delta^*}\dfrac{\overline{A}_{u,u}\overline{z}_u}{\overline{z}_u} = \dfrac{\overline{A}_{u,u}}{\delta^*} \end{cases}, \tag{43}$$

and

$$\begin{cases} Prob^P(2,1) + Prob^P(2,2) = \dfrac{1}{\delta^*}\dfrac{A_{2,1}z_1 + A_{2,2}z_2}{z_2} \\ \overline{Prob}^P(u,u) = \dfrac{1}{\delta^*}\dfrac{\overline{A}_{u,u}\overline{z}_u}{\overline{z}_u} = \dfrac{\overline{A}_{u,u}}{\delta^*} \end{cases}. \tag{44}$$

Because the natural probabilities of both systems agree, we must have $\overline{Prob}^P(u,u) = Prob^P(1,1) + Prob^P(1,2) = Prob^P(2,1) + Prob^P(2,2)$. Thus, in order to prove $A_{1,1} + A_{1,2} = A_{2,1} + A_{2,2}$, it suffices to show $z_1 = z_2$. Since the consistency also implies $Prob^P(1,3) = Probe^P(2,3) = \overline{Prob}^P(u,d)$, i.e., $\dfrac{A_{1,3}z_3}{\delta^* z_1} = \dfrac{A_{2,3}z_3}{\delta^* z_2}$, it is equivalent to showing $A_{1,3} = A_{2,3}$.

For simplicity, write the original A as $\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & j \end{pmatrix}$. In the three-state model, we have

$$\underbrace{\begin{pmatrix} A^2_{i,1} & A^2_{i,2} & A^2_{i,3} \\ A^3_{i,1} & A^3_{i,2} & A^3_{i,3} \\ A^4_{i,1} & A^4_{i,2} & A^4_{i,3} \end{pmatrix}}_{\mathcal{A}_{2\uparrow}} = \underbrace{\begin{pmatrix} A_{i,1} & A_{i,2} & A_{i,3} \\ A^2_{i,1} & A^2_{i,2} & A^2_{i,3} \\ A^3_{i,1} & A^3_{i,2} & A^3_{i,3} \end{pmatrix}}_{\mathcal{A}_{1\uparrow}} \underbrace{\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & j \end{pmatrix}}_{A},$$

and in the two-state model we get

$$\underbrace{\begin{pmatrix} A^2_{i,1} + A^2_{i,2} & A^2_{i,3} \\ A^3_{i,1} + A^3_{i,2} & A^3_{i,3} \end{pmatrix}}_{\overline{\mathcal{A}}_{2\uparrow}} = \underbrace{\begin{pmatrix} A_{i,1} + A_{i,2} & A_{i,3} \\ A^2_{i,1} + A^2_{i,2} & A^2_{i,3} \end{pmatrix}}_{\overline{\mathcal{A}}_{1\uparrow}} \underbrace{\begin{pmatrix} \overline{A}_{u,u} & \overline{A}_{u,d} \\ \overline{A}_{d,u} & \overline{A}_{d,d} \end{pmatrix}}_{\overline{A}}.$$

The consistency in consolidation yields $Prob^P(3,3) = \overline{Prob}^P(d,d)$, which implies $\overline{A}_{d,d} = j$. Therefore,

$$\begin{cases} A^2_{i,3} = \overline{A}_{u,d}(A_{i,1} + A_{i,2}) + jA_{i,3} \\ A^3_{i,3} = \overline{A}_{u,d}(A^2_{i,1} + A^2_{i,2}) + jA^2_{i,3} \end{cases}. \tag{45}$$

Without loss of generality, assume the current state $i = 1$, i.e., the first row of $\mathcal{A}_{1\uparrow}$ is the first row of A. From the first equation of (45) we get

$$\overline{A}_{u,d} = \frac{A^2_{i,3} - jA_{i,3}}{A_{i,1} + A_{i,2}} = \frac{ac + bf + cj - cj}{a + b} = \frac{ac + bf}{a + b},$$

and from the second equation

$$\overline{A}_{u,d} = \frac{A^3_{i,3} - jA^2_{i,3}}{A^2_{i,1} + A^2_{i,2}} = \frac{(a^2+bd+cg)c + (ab+be+ch)f + (ac+bf+cj)j - (ac+bf+cj)j}{(a^2+bd+cg) + (ab+be+ch)}$$
$$\equiv \frac{Xc+Yf}{X+Y}.$$

Thus, $\frac{ac+bf}{a+b} = \frac{Xc+Yf}{X+Y}$. By simplification, we obtain $c(aY-bX) = f(aY-bX)$. If $aY - bX \neq 0$, then we are done. Suppose $aY - bX = 0$. By the definition that $X = A^2_{i,1}$ and $Y = A^2_{i,2}$, we have

$$A_{i,1}A^2_{i,2} = A_{i,2}A^2_{i,1}. \tag{46}$$

Similar to (45), we have

$$A^4_{i,3} = \overline{A}_{u,d}(A^3_{i,1} + A^3_{i,2}) + jA^3_{i,3} \tag{47}$$

Therefore,

$$\overline{A}_{u,d} = \frac{A^4_{i,3} - jA^3_{i,3}}{A^3_{i,1} + A^3_{i,2}} = \frac{A^3_{i,1}c + A^3_{i,2}f + A^3_{i,3}j - A^3_{i,3}j}{A^3_{i,1} + A^3_{i,2}} \equiv \frac{X^*c + Y^*f}{X^* + Y^*}.$$

Thus, $\frac{Xc+Yf}{X+Y} = \frac{X^*c+Y^*f}{X^*+Y^*}$. By simplification, we get $c(XY^* - YX^*) = f(XY^* - YX^*)$. Again, suppose $XY^* - YX^* = 0$ because otherwise we would have $c = f$ and we are done then. If $XY^* - YX^* = 0$, by definition we get

$$A^2_{i,1}A^3_{i,2} = A^2_{i,2}A^3_{i,1}. \tag{48}$$

Together with (46), we arrive at

$$A^3_{i,1}A_{i,2} = A^3_{i,2}A_{i,1}. \tag{49}$$

Notice that the determinant of $\mathcal{A}_{1\uparrow}$ is

$$A_{i,1}[A^2_{i,2}A^3_{i,3} - A^2_{i,3}A^3_{i,2}] - A_{i,2}[A^2_{i,1}A^3_{i,3} - A^2_{i,3}A^3_{i,1}] + A_{i,3}[A^2_{i,1}A^3_{i,2} - A^2_{i,2}A^3_{i,1}].$$

Equations (46), (48), and (49) imply that the $\det(\mathcal{A}_{1\uparrow}) = 0$. Thus, the state price is over-redundant, which is a contradiction to unique recovery. Therefore, we must have $c = f$, i.e., $A_{1,3} = A_{2,3}$, which in turns yields $z_1 = z_2$. Hence, $A_{1,1} + A_{1,2} = A_{2,1} + A_{2,2}$. Together with $A_{1,3} = A_{2,3}$, we have proved the necessary condition for this special case. The necessity proof for general case is tedious and follows from a general matrix manipulation technique similar to the proof of Proposition 2. ■

A.2 Proof of OLS for $n = 2$ reducing to $\overline{n} = 1$

This proof uses the traditional method as analytic result can be achieved in this special case. Let the original 2×2 *AD* price matrix be A with eigenvalues δ_1 and δ_2. Denote the associated *left* eigenstates as w_1 and w_2. We can find twp scalars β_1 and β_2 such that $(A_{i,1}, A_{i,2}) = \beta_1 w_1 + \beta_2 w_2$. Since $\left(A^{t+1}_{i,1}, A^{t+1}_{i,2}\right) = \left(A^t_{i,1}, A^t_{i,2}\right) \times A$, we have that $\left(A^2_{i,1}, A^2_{i,2}\right) = \beta_1 w_1 A + \beta_2 w_2 A = \beta_1 \delta_1 w_1 + \beta_2 \delta_2 w_2$. Repeating this procedure, we can write the vector of state prices at any time t as

$$\left(A^t_{i,1}, A^t_{i,2}\right) = \beta_1 \delta_1^{t-1} w_1 + \beta_2 \delta_2^{t-1} w_2. \tag{50}$$

Written in matrix form, (50) becomes

$$\begin{pmatrix} \beta_1\delta_1 w_1 + \beta_2\delta_2 w_2 \\ \beta_1\delta_1^2 w_1 + \beta_2\delta_2^2 w_2 \\ \beta_1\delta_1^3 w_1 + \beta_2\delta_2^3 w_2 \\ \vdots \\ \beta_1\delta_1^h w_1 + \beta_2\delta_2^h w_2 \\ \beta_1\delta_1^{h+1} w_1 + \beta_2\delta_2^{h+1} w_2 \\ \vdots \\ \vdots \end{pmatrix} = \begin{pmatrix} \beta_1 w_1 + \beta_2 w_2 \\ \beta_1\delta_1 w_1 + \beta_2\delta_2 w_2 \\ \beta_1\delta_1^2 w_1 + \beta_2\delta_2^2 w_2 \\ \vdots \\ \beta_1\delta_1^{h-1} w_1 + \beta_2\delta_2^{h-1} w_2 \\ \beta_1\delta_1^h w_1 + \beta_2\delta_2^h w_2 \\ \vdots \\ \vdots \end{pmatrix} \times A$$

Define $w_1 \equiv w_1(1) + w_1(2)$, i.e., the sum of entries in w_1, and similarly $w_b = w_2(1) + w_2(2)$. Applying the consolidation procedure (12), we get

$$\underbrace{\begin{pmatrix} \beta_1\delta_1 w_a + \beta_2\delta_2 w_b \\ \beta_1\delta_1^2 w_a + \beta_2\delta_2^2 w_b \\ \beta_1\delta_1^3 w_a + \beta_2\delta_2^3 w_b \\ \vdots \\ \beta_1\delta_1^h w_a + \beta_2\delta_2^h w_b \\ \beta_1\delta_1^{h+1} w_a + \beta_2\delta_2^{h+1} w_b \\ \vdots \\ \vdots \end{pmatrix}}_{\overline{\mathcal{A}}_{2\uparrow}} = \underbrace{\begin{pmatrix} \beta_1 w_a + \beta_2 w_b \\ \beta_1\delta_1 w_a + \beta_2\delta_2 w_b \\ \beta_1\delta_1^2 w_a + \beta_2\delta_2^2 w_b \\ \vdots \\ \beta_1\delta_1^{h-1} w_a + \beta_2\delta_2^{h-1} w_b \\ \beta_1\delta_1^h w_a + \beta_2\delta_2^h w_b \\ \vdots \\ \vdots \end{pmatrix}}_{\overline{\mathcal{A}}_{1\uparrow}} \times \overline{\delta},$$

where $\overline{\delta}$ is the time preference for the consolidated one-state model, which, according to (8), is $\left(\overline{\mathcal{A}}_{1\uparrow}^T \overline{\mathcal{A}}_{1\uparrow}\right)^{-1} \overline{\mathcal{A}}_{1\uparrow}^T \overline{\mathcal{A}}_{2\uparrow}$. Expressing $\overline{\delta}$ explicitly, we have

$$\begin{aligned} \overline{\mathcal{A}}_{1\uparrow}^T \overline{\mathcal{A}}_{2\uparrow} &= (\beta_1\delta_1 w_a + \beta_2\delta_2 w_b)(\beta_1 w_a + \beta_2 w_b) + (\beta_1\delta_1^2 w_a + \beta_2\delta_2^2 w_b)(\beta_1\delta_1 w_a + \beta_2\delta_2 w_b) + \cdots \\ &= \beta_1^2 w_a^2 \delta_1 (1 + \delta^2 + \delta_1^4 + \cdots) + \beta_2^2 w_b^2 \delta_2 (1 + \delta_2^2 + \delta_2^4 + \cdots) \\ &\quad + \beta_1\beta_2\delta_2 w_a w_b (1 + \delta_1\delta_2 + \delta_1^2\delta_2^2 + \cdots) + \beta_1\beta_2\delta_1 w_a w_b (1 + \delta_1\delta_2 + \delta_1^2\delta_2^2 + \cdots) \\ &= \beta_1^2 w_a^2 \frac{\delta_1}{1-\delta_1^2} + \beta_2^2 w_b^2 \frac{\delta_2}{1-\delta_2^2} + \beta_1\beta_2 w_a w_b \frac{\delta_1}{1-\delta_1\delta_2} + \beta_1\beta_2 w_a w_b \frac{\delta_2}{1-\delta_1\delta_2}, \end{aligned}$$

and

$$\begin{aligned} \overline{\mathcal{A}}_{1\uparrow}^T \overline{\mathcal{A}}_{1\uparrow} &= (\beta_1 w_a + \beta_2 w_b)^2 + (\beta_1\delta_1 w_a + \beta_2\delta_2 w_b)^2 + (\beta_1\delta_1^2 w_a + \beta_2\delta_2^2 w_b)^2 + \cdots \\ &= \beta_1^2 w_a^2 (1 + \delta_1^2 + \delta_1^4 + \cdots) + \beta_2^2 w_b^2 (1 + \delta_2^2 + \delta_2^4 + \cdots) + 2\beta_1\beta_2 w_a w_b (1 + \delta_1\delta_2 + \delta_1^2\delta_2^2 + \cdots) \\ &= \beta_1^2 w_a^2 \frac{1}{1-\delta_1^2} + \beta_2^2 w_b^2 \frac{1}{1-\delta_2^2} + 2\beta_1\beta_2 w_a w_b \frac{1}{1-\delta_1\delta_2}. \end{aligned}$$

We can see that $\overline{\delta} = \left(\overline{\mathcal{A}}_{1\uparrow}^T \overline{\mathcal{A}}_{1\uparrow}\right)^{-1} \overline{\mathcal{A}}_{1\uparrow}^T \overline{\mathcal{A}}_{2\uparrow}$ can be neither δ_1 nor δ_2 unless $\delta_1 = \delta_2$, which is not possible for the 2×2 matrix A. Hence, this shows that OLS does not work for the case of $n = 2$ reducing to $\overline{n} = 1$. ■

B Proof of Proposition 3

In short-horizon limit, we examine the current price p of a state-contingent payoff H realized at $t+dt$ in either settings,

$$\begin{aligned}
\text{discrete setting} \quad : \quad p_{i,t} &= \textstyle\sum_j^n A_{ij}^{dt} H_j \\
\text{continuous setting} \quad : \quad p(x_t) &= E_t^Q\left[e^{-r(x_t)dt} H(x_{t+dt})\right] \\
&= H(x_t) + dt\left(x\mu_x^Q \tfrac{d}{dx} + \tfrac{1}{2}x^2\sigma_x^2 \tfrac{d^2}{dx^2} - r(x)\right) H(x_t) \\
&= \left[\mathbb{1} + dt\mathcal{D}^Q\right] H(x_t).
\end{aligned}$$

These pricing formulas imply the correspondence at short horizon (28). The precise mapping between A^{dt} and $\mathcal{D}^Q$ is given in (30).

In long-horizon limit, we start with a basic integration representation[37] for the pricing formula for a contingent payoff $H(X_T)$ realized at T, [38]

$$\begin{aligned}
p(x_t) = E_t^Q\left[e^{-\int_t^T r(x_s)ds} H(X_T)\right] &= H(X_t) + E_t^Q\left[\textstyle\int_t^T ds e^{-\int_t^s r(x_\tau)d\tau} \mathcal{D}^Q H(X_s)\right] \\
&= H(X_t) + \textstyle\int_t^T ds E_t^Q\left[e^{-\int_t^s r(x_\tau)d\tau} \mathcal{D}^Q H(X_s)\right].
\end{aligned}$$

Repeating this representation for the inner conditional expectation,[39] and substituting it back into the above pricing formula yield

$$p(x_t) = H(X_t) + \int_t^T ds \left\{\mathcal{D}^Q H(X_t) + \int_t^s dl \left[\left(\mathcal{D}^Q\right)^2 H(X_t) + \ldots\right]\right\} \equiv e^{\int_t^T ds \mathcal{D}^Q} H(X_t).$$

where the last exponential operator should be understood as the power series of $\mathcal{D}^Q$,

$$e^{\int^t ds \mathcal{D}^Q} H(X_t) \equiv \left\{\mathbb{1} + \int_t^T ds\mathcal{D}^Q + \frac{1}{2}\int_t^T ds \int_t^T dl \left(\mathcal{D}^Q\right)^2 + \ldots\right\} H(X_t). \tag{51}$$

On the other hand, the price can also be computed in discrete space setting using long-horizon AD prices matrix A^{T-t}, that is, $p(i,t) = \sum_j A_{ij}^{T-t} H_j$. The matching these prices, $p(x_t) \leftrightarrow p_{i,t}$, yields the long-horizon correspondence $A^{T-t} \leftrightarrow e^{\int_t^T ds\mathcal{D}^Q}$ of Proposition 3. The

[37] This representation is also known as Dynkin's formula.

[38] We assume standard regularity conditions to assure finite prices, which in turn allow for the interchange in the order of expectation and integration.

[39] Which explicitly is $E_t^Q\left[e^{-\int_t^s r(x_\tau)d\tau} \mathcal{D}^Q H(X_s)\right] = \mathcal{D}^Q H(X_t) + \int_t^s dl E_t^Q\left[e^{-\int_t^l r(x_k)dk} \left(\mathcal{D}^Q\right)^2 H(X_l)\right]$.

precise mapping of this correspondence is given (30). ■

C Perron-Frobenius and Krein-Rutman Theorems

C.1 Perron-Frobenius Theorem

The early results concerning matrices with positive entries were due to Oskar Perron (1907). Ferdinand Georg Frobenius (1912) extended the results to matrices that have non-negative entries and are irreducible (see definition below).

Theorem 1 (Perron's original version) *Let $A = [a_{ij}]$ be an $n \times n$ matrix whose entries are all positive. Then, there is a positive real number δ, called the* Perron root*, such that δ is an eigenvalue of A and any other eigenvalue λ is strictly smaller than δ in absolute value; there exists a right eigenstate $z = [z_1, \cdots, z_n]$ of A with eigenvalue δ whose components are all positive. Moreover, the eigenspace associated to δ is one-dimensional.*

Definition 2 *Let A be a square matrix, not necessarily positive. A is called* irreducible *if A^m is positive for some $m \in \mathbb{N}^+$.*

Theorem 3 (Frobenius's extended version) *Let A be an $n \times n$ non-negative and irreducible matrix with spectral radius $\rho(A) = \delta$. Then, δ is a positive real number and an eigenvalue of A; all components of the associated right eigenstate z are strictly positive. Moreover, the eigenspace associated to δ is one-dimensional.*

The key difference between the two versions is that in the former one δ is unique while eigenvalues with largest absolute value may not be unique in the latter one.

C.2 Krein-Rutman Theorem

This theorem is a generalization of Perron-Frobenius theorem to infinite-dimensional Banach spaces. (See Du, 2006)

Definition 4 *Let X be a Banach space. $K \subset X$ is called a* cone *if K is a closed convex set such that $\lambda K \subset K$ for all $\lambda \geq 0$ and $K \cap (-K) = \{0\}$; if the set $\{u - v : u, v \in K\}$ is dense in X, then K is called a* total cone*; if K has nonempty interior int(K), then it is called a* solid cone*.*

Theorem 5 *Let X be a Banach space, $K \subset X$ a total and solid cone, and $T : X \to X$ a compact linear operator which is strictly positive, i.e., $Tu \gg 0$ if $u > 0$. Assume the spectral radius $\delta = \rho(T)$ is positive. Then, δ is an eigenvalue of T with a positive eigenstate z and any other eigenvalue of T is strictly smaller than δ in absolute value. Moreover, δ is simple and $z \in int(K)$, and there is no other eigenvalue with a positive eigenstate.*

D Bakshi-Kapadia-Madan formula

In this section we present the formulas for risk-neutral dynamic $\{\mu^Q, \sigma\}$ derived in Bakshi, Kapadia, and Madan (2003) from option prices. The formulas are model-independent (they instead rely on no-abitrages). Also note that although the underlying state is the price S_t of traded stock, μ^Q is not risk-free rate simply because here the convention $\mu^Q \equiv E_t^Q\left[\log \frac{S(t+\tau)}{S(t)}\right]$ simply is not the short-term mean growth $\frac{1}{dt}E_t^Q\left[\frac{dS(t)}{S(t)}\right]$ of stock price. For simplicity, define S to be the $t+\tau$ period stock price, $q(S)$ to be the risk-neutral pricing density $q(t,\tau;S)$, and $H(S)$ to be the payoff of any contingent claim such that $\int_0^\infty \mid H(S) \mid q(S)dS < \infty$. Then the risk-neutral expectation of the payoff is

$$E_t^Q[H(S)] = \int_0^\infty H(S)q(S)dS. \tag{52}$$

A special case in Theorem 1 of Bakshi and Madan (2000) shows that the entire collection of twice continuously differentiable payoff functions, $H(S) \in \mathcal{C}^2$, can be spanned algebraically as in

$$\begin{aligned} H(S) = H(K) + (S-K)H_S(K) + \int_K^\infty H_{SS}(X)(S-X)^+ dX \\ + \int_0^K H_{SS}(X)(X-S)^+ dX, \end{aligned} \tag{53}$$

where K is the strike price and H_S and H_{SS} are first- and second-order derivatives of H with respect to S.

We obtain the price of any contingent claim by applying (52) to (53):

$$\begin{aligned} E_t^Q[e^{-r\tau}H(S)] = [H(K) - KH_S(K)]e^{-r\tau} + H_S(K)S(t) \\ + \int_K^\infty H_{SS}(X)C(t,\tau;X)dX + \int_0^K H_{SS}(X)P(t,\tau;X)dX, \end{aligned} \tag{54}$$

where $C(t,\tau;X)$ ($P(t,\tau;X)$) is the call (put) price with strike X and maturity τ.

Let the τ-period log return of stock price be $R(t,\tau) := \log[S(t+\tau)/S(t)]$. Define the volatility contract to have the payoff $H(S) = R(t,\tau)^2$ with a fair price of $V(t,\tau) = E_t^Q[e^{-r\tau}R(t,\tau)^2]$. We similarly define cubic and quartic contracts with payoffs $R(t,\tau)^3$ and $R(t,\tau)^4$ respectively. Then, Theorem 1 in Bakshi, Kapadia, and Madan (2003) yields the volatility of stock price as

$$\sigma = \sqrt{e^{r\tau}V(t,\tau) - \mu^Q(t,\tau)^2}, \tag{55}$$

where

$$\mu^Q(t,\tau) = E_t^Q\left[\log\frac{S(t+\tau)}{S(t)}\right] = e^{r\tau} - 1 - \frac{e^{r\tau}}{2}V(t,\tau) - \frac{e^{r\tau}}{6}W(t,\tau) - \frac{e^{r\tau}}{24}Y(t,\tau), \tag{56}$$

$$V(t,\tau) = \int_{S(t)}^{\infty}\frac{2\left(1-\log\left[\frac{X}{S(t)}\right]\right)}{X^2}C(t,\tau;X)dX + \int_0^{S(t)}\frac{2\left(1+\log\left[\frac{S(t)}{X}\right]\right)}{X^2}P(t,\tau;X)dX, \tag{57}$$

$$\begin{aligned} W(t,\tau) = &\int_{S(t)}^{\infty}\frac{6\log\left[\frac{X}{S(t)}\right] - 3\left(\log\left[\frac{X}{S(t)}\right]\right)^2}{X^2}C(t,\tau;X)dX \\ &- \int_0^{S(t)}\frac{6\log\left[\frac{S(t)}{X}\right] + 3\left(\log\left[\frac{S(t)}{X}\right]\right)^2}{X^2}P(t,\tau;X)dX, \end{aligned} \tag{58}$$

$$\begin{aligned} Y(t,\tau) = &\int_{S(t)}^{\infty}\frac{12\left(\log\left[\frac{X}{S(t)}\right]\right)^2 - 4\left(\log\left[\frac{X}{S(t)}\right]\right)^3}{X^2}C(t,\tau;X)dX \\ &+ \int_0^{S(t)}\frac{12\left(\log\left[\frac{S(t)}{X}\right]\right)^2 + 4\left(\log\left[\frac{S(t)}{X}\right]\right)^3}{X^2}P(t,\tau;X)dX. \end{aligned} \tag{59}$$

With the equations above, we obtain the risk-neutral dynamics $\{\mu^Q, \sigma\}$.

E Breeden-Litzenberger formula

Let the current underlying price be S_0. Define $p(S_0, S, T)$ to be the risk-neutral state price density at any tenor T. We can write the price of a call option with strike K and maturity

T as

$$C(S_0, K, T) = e^{-rT} \int_0^\infty (S - K)^+ p(S_0, S, T) dS = e^{-rT} \int_K^\infty (S - K) p(S_0, S, T) dS. \quad (60)$$

We differentiate equation (60) twice with respect to K to obtain the Breeden-Litzenberger formula:

$$p(S_0, S, T) = \frac{\partial^2 C(S_0, K, T)}{\partial K^2} e^{rT}. \quad (61)$$