Recovery and Consistency^{*}

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Abstract

Recovery, the process of uniquely determining market's belief, time and risk preferences from asset prices, requires a subjective state-space specification of the underlying economy that is not observed before the recovery is implemented. Different subjective input specifications lead to different recovery results that, albeit unique under the respective specifications, are almost surely inconsistent with each other. This consistency issue prevails universally in the original, generalized, and perturbative recovery approaches, and is not resolved by the sophistication of the input specification, perfect (error-free and infinite) price data, or the enforcement of required recovery assumptions. The direction of the recovery inconsistency, or the signed difference between the recovered and underlying quantities, is influenced by both the probability and marginal utility distributions of the underlying economic states such as the presence of a rare disaster state. The consistency requirement highlights a new and general challenge for the recovery paradigm.

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1 Introduction

The recovery concept in asset pricing sets a highly relevant and challenging task of separately identifying the market's risk and time preferences and belief about future states of the economy from asset prices. The concept is based on a fundamental no-arbitrage pricing result that, in equilibrium, an asset's price is equal to the expected discounted value of the asset's future cash flows, where the expectation and discounting are characterized respectively by market's belief and preferences toward economic uncertainties and risks in the cash flows. While being highly relevant as a direct asset-based estimation of the market's important characteristics, the recovery paradigm's principal challenge is how to unambiguously disentangle the market's preferences and belief using only asset prices.

Ross (2015)'s Recovery Theorem addresses this challenge by identifying sufficient conditions on cash flow and pricing dynamics, under which today's price data of assets across different maturities suffice to simultaneously and uniquely determine the market's preferences and probability distribution of future states of the economy. While the Recovery Theorem is elegant, its applications and merits depend on two key aspects, namely, the empirical content of the theorem's underlying assumptions and the theorem's implementability in practice. The current paper examines the recovery's implementability aspect and demonstrates a robust recovery consistency issue. This consistency issue prevails in implementing both the original and generalized versions of the Recovery Theorem, independent of the empirical content of the theorem's underlying assumptions. A recovery paradigm that yields consistent outcomes hence remains challenging and elusive.

While the recovery process requires a state space specification for the underlying market model to start with, this specification is not observed prior to the process implementation and so is a subjective input in the recovery. The recovery consistency then is a natural requirement that the results recovered with different input specifications be mutually consistent because these results pertain to the same underlying market model. Integrating the consistency concept into the recovery framework motivates and informs three main findings of the paper. First, we establish a necessary and sufficient condition for the recovery consistency, whose strictness affirms the prohibitive nature of a consistent recovery. Second, we identify the origin of the recovery consistency issue with an inevitable tradeoff between the sophistication of the input specification and a non-linear error buildup of the recovery process, implying that a finer state space specification does not resolve the consistency issue in the recovery. Third, we identify the direction (i.e., overshooting or undershooting) of the recovery inconsistency and the responsible economic features in a model calibrated to the U.S. economy.

Both the conceptual and practical aspects are crucial for the Recovery Theorem and its applications. Conceptually, assuming a time-separable preference and time-homogeneous (or, Markovian) state transition dynamics, the theorem establishes the dominant and unique eigenvalue and eigenvector of the one-period Arrow-Debreu (AD) price matrix **A** respectively as the time discount factor and vector of (inverse) marginal utilities of the underlying market. Practically, since **A** is not fully observed from price data, the theorem's applicability centers on how to obtain this matrix.¹ It is this important quest for a consistent determination of the one-period AD price matrix that elucidates the aforementioned findings most directly and intuitively.

First, albeit not fully observed, the one-period AD price matrix **A** can be implied from a procedure involving the current prices of longer-term AD assets initiated on the current state today but with various maturities. Specifically, when a subjective input specification of Sstates is adopted, this procedure implies a $S \times S$ matrix **A** needed for the recovery. Another subjective choice of \overline{S} states (say, a coarser specification, $\overline{S} < S$) implies a corresponding $\overline{S} \times \overline{S}$ AD price matrix \overline{A} . A key issue for this procedure is that the market information

¹The AD price matrix **A** contains the prices of all one-period AD assets A_{ij} , where *i* denotes the state in which the AD contract is initiated and *j* the next-period contingent state in which the contract pays off. On any day, one can only observe the current prices of AD contracts initiated on a single current state (of that day). Hence, the entire AD price matrix **A** is not fully and directly observed on any day.

contained in the fine AD asset contracts and prices associated with S fine states is irreversibly lost in the coarser AD asset contracts and prices associated with \overline{S} coarser states. The loss of information then results in irreconcilable market's characteristics recovered from **A** and $\overline{\mathbf{A}}$, except in a special premise of the underlying market model. Our paper's necessary and sufficient condition for the recovery consistency identifies and quantifies this special premise. Namely, if and only if all fine states (among S states) that belong to a coarser state (among \overline{S} states) are associated with identical characteristics (marginal utilities and probabilities), then the recovery results associated with S and \overline{S} states are consistent with each other. Intuitively, this condition is what is precisely needed to prevent an inadvertent loss of information in picking a subjective specification. Evidently, the condition is highly restrictive, indicating the prevalence of the recovery consistency issue.

Second, the recovery consistency issue originates endogenously from the determination of the implied AD price matrix **A**. On one hand, a more sophisticated specification (a larger number S of specified states) offers higher flexibility to accommodate the underlying market model. On the other hand, it requires a larger recursive equation system to solve for $S \times S$ matrix A, involving the pricing equations of longer-term AD assets maturing in S+1 different periods. As a result, even a small deviation between the underlying and input specifications (thanks to a good approximation associated with a large S) reverberates recursively (commensurately with the large S) in the recovery system, amplifies the approximation error's buildup and results in the recovery inconsistencies. In the discrete setting (i.e., finite S), this tradeoff is demonstrated analytically using a perturbative analysis. In the continuous setting (i.e., the infinitesimal limit of state space grid partition and unbounded S), this tradeoff is revealed in spurious negative AD asset prices (spurious arbitrage opportunities) and hence recovery inconsistencies. This persistent tradeoff signifies that the sophistication of the input specification and data quality do not alleviate the recovery consistency issue unless the underlying market model satisfies the strong necessary and sufficient condition discussed earlier.

Third, the direction of inconsistencies in the recovery results depends on not only the input specification but also the economic features of the underlying market model. To understand these features and the resulting inconsistency direction, we first calibrate the underlying market model to the U.S. economy, then implement and analyze its recovered version. The calibrated underlying model features a rare disaster state, which stylizes the U.S. economy in that this state is associated with an elevated marginal utility but exceedingly small probability.² To illustrate, assume that current state is a normal state but the input (subjective) specification coarsely and unknowingly confounds it with the underlying rare disaster state. This tangling of the normal (current) and the disaster states increases the perceived current marginal utility while also moderating the perceived magnitude of the disaster in the future (compared to their actual counterparts in the underlying market model). As a result, the insurances against (i.e., the AD assets paying in) the adverse (confounded) state are perceived to be cheaper, which translates into an undershooting of the recovered probability to remain in the current state (and a overshooting of the recovered probability to transit to another state), compared to the actual counterpart probabilities in the underlying model.

The standard improvement in the risk-neutral option pricing and the associated volatility surface determination with the refinement of the state space grid does not apply in the Recovery Theorem. This is because the option pricing concerns only contracts and prices initiated on the current state today. Whereas, the Recovery Theorem aims to determine the transition probabilities between any two possible states, hence requires the implied prices of AD assets initiated on states different from the current one. It is this implication of the full AD price matrix that eludes the standard improvement associated with the state space grid refinement. Furthermore, while a unique recovery result is obtained per a subjective input specification, this uniqueness does not preclude the inconsistencies across the results

²That is, concerning the rare disaster state, the pricing effect of its high marginal utility is dominated by its small likelihood. As a result, current prices of insurances against (i.e., AD assets paying off in) the rare disaster state remain moderate.

recovered associated with different input specifications in practice. These features highlight the prevalence and relevance of the consistency issue in the recovery paradigm.

Related literature: There has been a long-standing interest in the recovery of market's belief, risk and time preferences from observed asset prices, given that a unambiguous disentanglement between these characteristics is non-trivial. The risk-neutral approach is content with conditioning on a (fictitious) null risk aversion and recovers the associated risk-neutral belief from option prices as in Breeden and Litzenberger (1978). The advent of Ross (2015)'s Recovery Theorem, which conceptually formulates sufficient conditions needed to disentangle market's belief and risk preference, has renewed recent interests in the recovery. The empirical performance of the recovery based on this theorem remains elusive. A literature assesses the empirical content of the Recovery Theorem's assumptions. Borovička et al. (2016) and Hansen and Scheinkman (2017) relate the theorem's assumptions to the recoverability of only the transitory component of Alvarez and Jermann (2005)'s SDF growth decomposition. Bakshi et al. (2018), Qin et al. (2018) and Jackwerth and Menner (2020) examine and provide evidence for counterfactual implications of these assumptions on asset prices. Taking all assumptions of the Recovery Theorem as given, our paper complements this literature by focusing on the implementability and consistency aspects of the recovery approach. Altogether, these findings on the recovery paradigm's challenges indicate two (non mutually exclusive) possibilities. First, surveys remain the direct and informative channel to learn about market's belief and their rich and complex contingent investment decisions Giglio et al. (2022). Second, under some weak assumptions on market's behaviors and without invoking specific parametric models, asset price data can only provide us with some useful bounds on market's expectations as demonstrated by, e.g., Martin (2017) and Gormsen and Koijen (2020).

Another literature aims to generalize the Recovery Theorem by extending its premise or relaxing its restrictive assumptions. Carr and Yu (2012) derive the recovery in a continuous setting given bounded underlying stochastic state variables. Walden (2017) generalizes the recovery to settings with an unbounded support of the state dynamics. Qin and Linetsky (2016) extend Ross's recovery to continuous-time Markov processes. Huang and Shaliastovich (2014) consider the recovery in a recursive utility framework. Dillschneider and Maurer (2019) discuss the Perron-Frobenius operator theory in recovery. Martin and Ross (2019) discuss the relationship between the recovered time preference and the unconditional expected return on long-maturity bonds. Jensen et al. (2019) propose a generalized recovery approach by relaxing the time homogeneous assumption and accommodating a growing state space. Pazarbasi et al. (2023) provide a bound on the dispersion of beliefs in a sentimental recovery setting for both complete and incomplete markets. Related to and built on this literature, our paper examines the recovery consistency issue for both original, generalized, and approximate (best-fit) recovery approaches. Several papers examine the sensitivity and stability issue related to the recovery process. Walden (2017) shows that a recovery process can be unstable under a small perturbation in state prices. Dubynskiy and Goldstein (2013) consider the effect of boundary conditions on the recovery process's stability. In a related principal eigenproblem in asset pricing, Borovička and Stachurski (2020) derive a necessary and sufficient condition for the existence of a unique and stable value function for a class of recursive utilities. Our paper formalizes the recovery consistency notion in terms of how sensitive the recovery results are under changes in the input specification.

The paper is organized in accordance with the main findings. Section 2 introduces the concept of, and presents a necessary and sufficient condition for, the consistency in the recovery process. Section 3 explains intuitively and demonstrates quantitatively the origin of the recovery consistency issue in terms of a tradeoff between the specification's sophistication and the error buildup of the recovery process. Section 4 examines the direction of, and identify the economic features responsible for, the recovery inconsistency in a model calibrated to the U.S. economy. Section 5 concludes. Appendices A, B, and C present technical derivations, calibration details, and extensions of the paper's findings.

2 Recovery and Consistency

We describe Ross (2015)'s basic framework and implementation procedure of recovery (Section 2.1), and discuss the consistency concept for recovery (Section 2.2).

2.1 Recovery

The basic recovery framework can be described in either discrete or continuous settings, offering an integral perspective on the recovery and its consistency issue in different limits of the state space specification.

Discrete Setting

The basic market model is in a discrete setting, in which the state and time are denoted by (i,t), with $\{i\} \in S \equiv \{1,\ldots,S\}$ and $t \in \{0,\ldots,T\}$. We assume that the financial market is complete and free of arbitrage opportunities. As a result, prices of financial assets are determined by the no-arbitrage pricing principle, which features a unique stochastic discount factor (SDF). Let $M_{t,t+1}(i,j)$ denote the growth of the SDF associated with the transition from state time (i,t) to (j,t+1). Given a complete financial market, the SDF can be identified with the marginal utility of the representative agent in the market model. The recovery process aims to uniquely determine the state probability distribution in the physical measure and the representative agent's preference from asset prices. Such a recovery process relies on the following two assumptions.

Assumption A1. The preference is time-separable, or the SDF growth has the following functional form: $M_{t,t+1}(i,j) = \delta \frac{M_j}{M_i}, \forall t \in \{0,\ldots,T-1\}, \forall \{i\}, \{j\} \in S$, where δ is a constant parameter and M_i is a function of state $\{i\}$ (but not time).

Assumption A2. The state transition dynamics are time-homogeneous, or the transition probabilities in the physical measure are time-independent: $p_{t,t+1}(i,j) = p_{i,j}, \forall t \in \{0, \ldots, T-1\}, \forall \{i\}, \{j\} \in S$.

The first assumption can be motivated from an economic perspective of a representative agent's time-separable utility function, and the second assumption from a Markovian perspective of the state transition dynamics. In this picture, δ and M_i respectively quantifies the representative agent's time discount factor and marginal utility. Under these assumptions, the unique recovery of preference and state probability distribution can be derived as follows.

Consider a state-contingent financial asset that pays an amount equal to the inverse of marginal utility, $x_j \equiv \frac{1}{M_j}$, in state $\{j\}$ next period. If the current period's state is $\{i\}$, the no-arbitrage current price $P_{it}\left(\frac{1}{M}\right)$ of this asset is

$$P_{it}\left(\frac{1}{M}\right) = E_{it}\left[M_{t,t+1}(i,j)\frac{1}{M_j}\right] = \delta \frac{1}{M_i}, \quad \text{or} \quad E_{it}\left[M_{t,t+1}(i,j)x_j\right] = \delta x_i, \quad \forall \{j\} \in \{1,\dots,S\},$$

$$(1)$$

where we have used $M_{t,t+1}(i,j) = \delta \frac{M_j}{M_i}$ (Assumption A1). Alternatively, the financial asset above can also be priced as a portfolio with weight x_j in the respective one-period Arrow-Debreu (AD) asset A_{ij} offering a unit payoff if (and only if) the next period's state is $\{j\}$, where $\{j\} \in \{1, \ldots, S\}$. Hence, the current price $P_{it}\left(\frac{1}{M}\right)$ (1) can also be expressed as,³ $P_{it}\left(\frac{1}{M}\right) = \sum_{\{j\}=1}^{S} A_{ij}x_j$. Identifying this price with (1) implies that

$$\sum_{\{j\}\in\mathcal{S}} A_{ij}x_j = \delta x_i, \quad \text{or} \quad \mathbf{A}\mathbf{x} = \delta \mathbf{x}, \quad \text{with } x_i = \frac{1}{M_i}, \, \forall \{i\} \in \{1, \dots, S\},$$
(2)

where **A** denotes the $S \times S$ one-period AD price matrix (whose entries are AD prices A_{ij}) and **x** denotes the $S \times 1$ vector of inverse marginal utilities. Throughout, we employ bold characters to denote matrix and vector quantities. Equation system (2) provides a conceptual interpretation that when preferences are time-separable (Assumption A1), the time discount factor and the inverse of marginal utilities are respectively the eigenvalue and eigenvector of the one-period AD price matrix **A**. In the absence of arbitrage opportunities, all entries of

³For notational conveniences, we also use the notation A_{ij} to denote the current price at (i, t) of the AD asset that offers a unit payoff at (j, t + 1) and zero otherwise.

AD price matrix **A** are strictly positive. Matrix **A** therefore has a unique eigenvector **x** whose elements are strictly positive per Perron-Frobenius theorem. This eigenvector is associated with the largest and positive eigenvalue δ . Since marginal utilities and time discount factor are also strictly positive, a unique recovery is achieved by identifying the Perron-Frobenius dominant eigenvalue and eigenvector of the AD price matrix with the time discount factor and marginal utilities respectively. The transition probability $p_{t,t+1}(i,j)$ from (i,t) to (j,t+1)then follows from the pricing equation for AD assets in the physical measure,

$$A_{ij} = E_{it} \left[M_{t,t+1}(i,s) \mathbb{1}_j(s) \right] \Longrightarrow p_{t,t+1}(i,j) = \delta^{-1} A_{ij} \frac{M_i}{M_j} = \delta^{-1} A_{ij} \frac{x_j}{x_i}, \tag{3}$$

where the indicator function $\mathbb{1}_{j}(s)$ denotes the payoff of AD asset A_{ij} .

At any time t, we do not observe the price A_{kj} of AD assets that are initiated on states k different from the current state at t, i.e., we do not directly observe the entire $S \times S$ one-period AD price matrix **A** in the financial market. It is the Assumption A2 on the time-homogeneity of the state transition dynamics that help to imply this matrix **A** for the recovery process. Specifically, let $A_{\tau;ij}$ be the current price of τ -period AD asset initiated on the current state $\{i\}$ that offers a unit payoff if the state in τ periods is $\{j\}$ and zero otherwise. Prices $\{A_{\tau+1;ij}\}, \{j\} \in S$, of $(\tau + 1)$ -period AD assets initiated on the same current state $\{i\}$ are obtained recursively by rolling τ -period AD assets $\{A_{\tau;ij}\}$ one more period. The time invariance of the one-period AD price matrix (Assumption A2) allows for stacking these pricing equations recursively for $\tau \in \{1, \ldots, T\}$,

$$\begin{bmatrix} A_{2;i1} & \dots & A_{2;iS} \\ A_{3;i1} & \dots & A_{3;iS} \\ \vdots & \dots & \vdots \\ A_{\tau+1;i1} & \dots & A_{\tau+1;iS} \end{bmatrix} = \underbrace{\begin{bmatrix} A_{i1} & \dots & A_{iS} \\ A_{2;i1} & \dots & A_{2;iS} \\ \vdots & \dots & \vdots \\ A_{\tau;i1} & \dots & A_{\tau;iS} \end{bmatrix}}_{\equiv \mathbf{A}_{\tau}} \times \underbrace{\begin{bmatrix} A_{11} & \dots & A_{1S} \\ \vdots & \ddots & \vdots \\ A_{S1} & \dots & A_{SS} \end{bmatrix}}_{\equiv \mathbf{A}}, \quad (4)$$

where the one-period AD price A_{ij} is the abbreviated version of the full notation $A_{1;ij}$, $\forall \{i\}, \{j\}$. By definition, both matrices \mathbf{A}_{τ} and $\mathbf{A}_{\tau+1}$ record price data of AD assets initiated on the same current (today) state *i*, and hence are observable (today) in the financial market. When the price data of non-redundant AD assets maturing across S + 1 different periods (or maturities $\tau \in \{1, \ldots, S\}$ and $\tau + 1 \in \{2, \ldots, S+1\}$ in (4)) is collected, linear equations (4) is a just-identified system that solves for the one-period AD price matrix **A** needed in the recovery process (2). The recovery consistency issue is demonstrated and discussed in Appendix C.1 for a generalized recovery setting that relaxes Assumption A2 and in Appendix C.2 for a best-fit recovery setting that utilizes available price data that are more than required in a just-identified recovery system (4). Next, we briefly discuss the recovery process in the continuous setting, which helps to illustrate the recovery consistency issue at the infinitesimal limit of the discrete setting as discussed in the next section.

Continuous Setting

Let the market model be driven by a stochastic state variable y_t in continuous time. For ease of exposition, we assume that the state variable y_t is one-dimensional diffusion process

$$\frac{dy_{t+dt}}{y_t} = \frac{y_{t+dt} - y_t}{y_t} = \mu_y dt + \sigma_y dB_t = \mu_y^Q dt + \sigma_y dB_t^Q, \tag{5}$$

where B_t and B_t^Q are standard Brownian motions in the physical and risk-neutral measures respectively. The drift coefficients μ_y , μ_y^Q and volatility σ_y are subject to the recovery process and can be state-dependent, in which case they are processes adapted to the natural filtration generated by B_t and B_t^Q .⁴ As the state variable, y_t drives the SDF $M(y_t)$ and all other equilibrium quantities of the market model.

The recovery process in the continuous setting starts with identifying the pricing operator that is the counterpart to the AD price matrix in discrete setting. Let $A_{dt,ij}$ denote the

⁴In special cases in which y is the price of traded assets such as equities or equity indexes, the risk-neutral drift coefficient coincides with the risk-free rate, $\mu_y^Q = r(y)$.

current price of the AD asset offering a unit payment only if the state at an infinitesimal period $\tau = t + dt$ is $\{j\}$. Consider the contingent asset (1) whose payoff at time τ equals the inverse of the marginal utility, $x(y_{\tau}) \equiv \frac{1}{M(y_{\tau})}$. Comparing the pricing of this contingent asset using the risk-neutral measure with the pricing equation (1) using AD assets maps the discrete and continuous pricing operations in short and long time horizons,⁵

$$\begin{cases} \mathbf{A}_{dt} \longleftrightarrow \mathbb{1} + dt \mathcal{D}^{Q}, \\ \mathbf{A}_{t} \longleftrightarrow \exp\left(\int^{t} ds \mathcal{D}^{Q}\right), \end{cases} \quad \text{where} \quad \mathcal{D}^{Q} \equiv y \mu_{y}^{Q} \frac{d}{dy} + \frac{1}{2} y^{2} \sigma_{y}^{2} \frac{d^{2}}{dy^{2}} - r(y), \qquad (6) \end{cases}$$

where \mathcal{D}^Q is the infinitesimal operator in the risk-neutral measure, r(y) is the short-term interest rate process, and the exponential operator $\exp\left(\int^t ds \mathcal{D}^Q\right)$ is understood as the power series of \mathcal{D}^Q . Because the risk-neutral parameters $\{\mu_y^Q, \sigma_y\}$ can be estimated from option price data as in Breeden and Litzenberger (1978)'s semi-parametric approach, the risk-neutral operator \mathcal{D}^Q can also be constructed from price data.⁶

Since the infinitesimal operator \mathcal{D}^Q plays the preeminent role of the AD matrix **A** in pricing traded assets, it also replaces **A** (2) in the continuous-setting recovery process. Specifically, at short-term horizons, the recovery eigenvalue problem (2) reduces to $\mathbf{A}_{dt}\mathbf{x} = \delta^{dt}\mathbf{x}$. In the limit of infinitesimal (continuous) time interval $dt \to 0$, an application of the mapping (6) yields a continuous-time expression for this eigenvalue problem (see also Footnote 5)

$$\mathbf{A}_{dt}\mathbf{x} = \delta^{dt}\mathbf{x} \longleftrightarrow \left(\mathbb{1} + dt\mathcal{D}^Q\right)\mathbf{x}(y) = \left(\mathbb{1} + dt\log\delta\right)\mathbf{x}(y)$$

⁵First, in the infinitesimal period, the risk-neutral pricing of the contingent asset paying $x_{t+dt} = \frac{1}{M_{t+dt}}$ (1) is $E_t^Q \left[e^{-r_t dt} x_{t+dt} \right] = \delta^{dt} x_t$. To order dt (using the state variable process (5) and the Feynman-Kac differential representation for the risk-neutral expectation on the right hand side), this equation becomes $\left\{ 1 + \left(y \mu_y^Q \frac{d}{dy} + \frac{1}{2} y^2 \sigma_y^2 \frac{d^2}{dy^2} - r \right) dt \right\} x_t = \{ 1 + (\log \delta) dt \} x_t$. Second, in terms AD assets, the pricing of the same contingent asset is $\mathbf{A}_{dt} x_t = \delta^{dt} x_t$. The identification of these two pricing equations gives rise to the mapping in (6) and the expression for \mathcal{D}^Q therein.

⁶Bakshi et al. (2003) derive model-independent analytical formulas for first four risk-neutral moments of state variables in term of integrals of option prices. Dubynskiy and Goldstein (2013) show that price P(dy) of contract of payoff dy identifies μ_y^Q , and price $P(dy^2)$ of contract of payoff $(dy)^2$ identifies σ_y^2 . Large and discontinuous changes in state variable y can also be modeled by incorporating jump dynamics into the infinitesimal operator \mathcal{D}^Q .

or equivalently,
$$\mathcal{D}^Q \mathbf{x}(y) = -\rho \mathbf{x}(y)$$
, where: $\rho = -\log \delta$, $\delta = e^{-\rho}$. (7)

Note that ρ is the time discount rate in the continuous time. Using the expression (6) for the infinitesimal operator, the last equation is a differential equation in continuous setting

$$\frac{1}{2}y^2\sigma_y^2\frac{d^2\mathbf{x}(y)}{dy^2} + y\mu_y^Q\frac{d\mathbf{x}(y)}{dy} - r(y)\mathbf{x}(y) = -\rho\mathbf{x}(y).$$
(8)

Carr and Yu (2012) show that this differential equation has a *unique* positive solution $\mathbf{x}(y)$ (and the associated discount rate ρ) when (i) the state variable y is a bounded diffusion process, and (ii) appropriate Sturm-Liouville boundary conditions are imposed on the boundary of the support of y. In such a premise, Equation (8) presents the recovery equation in the continuous setting. Proceeding from short- to long-term horizons via the integral operator (6) in the continuous setting corresponds to rolling AD assets over one more period recursively (4) in the discrete setting and offers a unified perspective on the recovery process. Section 2.2 below discusses the recovery consistency requirement and employs the continuous setting as a limit to illustrate a robust recovery consistency issue.

2.2 Consistency

The recovery process requires a specification of the state space, e.g., the number S of states in the discrete setting (2). Since such a specification is not observed prior to the recovery process, it is subject to the discretion of the analyst who implements the recovery. It is then important to understand the impact of a subjective state space specification on the recovery results. Specifically, the recovery consistency poses the question on whether the two different sets of recovery results, recovered respectively and uniquely in two different subjective state space specifications, are consistent with each other. The current section formalizes the consistency notion in recovery and analyzes it in both discrete and continuous settings.

Discrete Setting

A thought experiment helps to explain the important role of the state space specification in a consistent recovery process. Since the number S of states is not observed, consider two analysts adopting two different and subjective (input) state space specifications, S = $\{1, \ldots, S\}$ of S states and $\overline{S} = \{\overline{1}, \ldots, \overline{S}\}$ of \overline{S} states. The thought experiment starts with mapping and consolidating the two input state space specifications to guide a comparison of the recovery results associated with these specifications.

Consolidation: Without loss of generality, let $\overline{S} < S$. We refer to S as the original specification and \overline{S} as the consolidated specification of the state space, which are presumed by two different analysts recovering the same underlying market model.⁷ A state $\{\overline{j}\} \in \overline{S}$ in the consolidated specification is referred to either as (i) a single state if it is identical to an original state $\{\overline{j}\} \equiv \{j\}$, or (ii) a coupled state if it is composed of multiple original states $\{\overline{j}\} \supset \{j\}$. Throughout, we employ an overbar to denote quantities associated with the consolidated specification \overline{S} . A simple example illustrates the mapping and consolidation of the two state space specifications.

Example 1 (State Space Specifications and Consolidation) The first (original) state space specification has three states $S = \{1, 2, 3\}$, and the second (consolidated) has two states $\overline{S} = \{\overline{1}, \overline{2}\}$. The first consolidated state is identical to the first original state, $\{\overline{1}\} = \{1\}$, *i.e.*, $\{\overline{1}\}$ is a single state. The second consolidated state is composed of the two remaining original states, $\{\overline{2}\} = \{2, 3\}$, *i.e.*, $\{\overline{2}\}$ is a coupled state.

The comparison and reconciliation of the recovery results proceed differently for single and coupled states. For a single state, we directly compare the recovery results in S and \overline{S} . For a coupled state, a consolidation process is needed for the comparison. We first aggregate the recovery results over all states in specification S that correspond to the coupled state in \overline{S}

⁷Hence, the references of the *original* and *consolidated* specifications are purely conventional since the analysts do not observe the true state space specification.



Figure 1: An illustration of different state space specifications S (left panel) and \overline{S} (right panel), and the consolidation scheme that relates these specifications in Example 1.

before making the comparison. For the specific Example 1, the consolidation and comparison are as follows. We directly compare the recovery results (i.e., marginal utility and transition probabilities) concerning state {1} in S and the single { $\overline{1}$ } in \overline{S} , and aggregate and compare the recovery results concerning states {2,3} in S and the coupled state { $\overline{2}$ } in \overline{S} .

Consistency conditions: Given that the same underlying (objective) model drives asset prices in the market, recovery results obtained by analysts with different subjective specifications are subject to the consistency requirement so that these recovery results consistently pertain to the same underlying market model. Intuitively, the recovery consistency requirement amounts to recovering identical values for (i) time discount factors, (ii) marginal utilities in single states, and (iii) probabilities for transitions among single states (and coupled states, after the appropriate consolidation is performed). When the consistency conditions are violated, the two sets of recovered characteristics are incompatible, so that at least one of them is also incompatible with the set of objective characteristics of the underlying market model. The consistency conditions present falsification criteria for the state space specification employed in the recovery process.

To illustrate the formulation of the consistency conditions, suppose that $\{1\}$ is the current underlying state in Example 1. Since this is a single state, $\{1\} \equiv \{\overline{1}\}$ (Figure 1), both analysts correctly perceive this same state as the current state in their respective specifications S and \overline{S} . The consistency conditions for the time discount factor and transition probabilities recovered by the analysts are direct,

$$\overline{\delta} = \delta, \qquad \overline{p}_{t,t+1}(\overline{1},\overline{1}) = p_{t,t+1}(1,1), \qquad \overline{p}_{t,t+1}(\overline{1},\overline{2}) = p_{t,t+1}(1,2) + p_{t,t+1}(1,3), \quad \forall t.$$
(9)

The consistency conditions for the recovered marginal utilities are indirect, i.e., being implied from the pricing of traded assets. The two analysts observe AD asset prices traded in the financial market and interpret these contractual prices in accordance with their subjective specifications. As their perceived current states are identical ($\{\overline{1}\} = \{1\}$) in the current illustration, they interpret an identical AD price contracted on these states $\overline{A}_{\overline{1}\overline{1}} = A_{11}$. But as their non-current states are not identical ($\{\overline{2}\} \equiv \{2,3\} \neq \{2\}, \{3\}$), they interpret different AD prices contracted on these states. Specifically, these prices are related as $\overline{A}_{\overline{1}\overline{2}} = A_{12} + A_{13}$. Substituting AD prices from the pricing equation (3) into this AD asset price relation (and using the consistency conditions (9)) implies the consistency condition for the recovered marginal utilities,

$$\frac{\overline{M}_{\overline{2}}}{\overline{M}_{\overline{1}}} = \frac{\frac{M_2}{M_1} p_{t,t+1}(1,2) + \frac{M_3}{M_1} p_{t,t+1}(1,3)}{p_{t,t+1}(1,2) + p_{t,t+1}(1,3)}.$$
(10)

Appendix A (Equations (62), (67)) formulates general consistency conditions for Ross's recovery approach, and Appendix C.1 (Equations (109), (115)) for the generalized recovery approach.

Continuous Setting

The recovery process in the continuous setting amounts to determining the unique bound state associated with the eigenproblem (8). Since an analytical solution to (8) does not exist in general, the recovery process involves discretizing the continuous state space to construct a numerical solution. Discretizing the state space not only yields a more explicit mapping between the AD price matrix **A** and the infinitesimal operator \mathcal{D}^Q (6), but also sheds light on the recovery consistency at the infinitesimal limit of the state space specification in the discrete setting. To discretize the state space, we consider a finite difference representation of the infinitesimal operator,

$$\mathcal{D}^{Q}\mathbf{x}(y) = -r(y)\mathbf{x}(y) + \mu_{y}^{Q}y\frac{\mathbf{x}(y+dy) - \mathbf{x}(y-dy)}{2dy} + \frac{1}{2}\sigma_{y}^{2}y^{2}\frac{\mathbf{x}(y+dy) - 2\mathbf{x}(y) + \mathbf{x}(y-dy)}{(dy)^{2}}.$$
(11)

This representation, together with the correspondence $\mathbf{A}_{dt} \longleftrightarrow (\mathbb{1} + dt \mathcal{D}^Q)$ (7), gives rise to a limiting (finite-difference) expression for the AD matrix associated with horizon dt,

$$\mathbf{A}_{dt} = \begin{pmatrix} X & Y & 0 & 0 & \dots & 0 & 0 & 0 \\ Z & X & Y & 0 & \dots & 0 & 0 & 0 \\ 0 & Z & X & Y & \dots & 0 & 0 & 0 & 0 \\ \vdots & & \ddots & & & \vdots & \vdots \\ 0 & 0 & 0 & \dots & Z & X & Y & 0 \\ 0 & 0 & 0 & \dots & 0 & Z & X & Y \\ 0 & 0 & 0 & \dots & 0 & 0 & Z & X \end{pmatrix}, \quad \text{where} \quad \begin{cases} X \equiv 1 - r(y)dt - \sigma_y^2 y^2 \frac{dt}{(dy)^2} \\ Y \equiv \frac{1}{2}\sigma_y^2 y^2 \frac{dt}{(dy)^2} + \frac{1}{2}\mu_y^Q y \frac{dt}{dy} \\ Z \equiv \frac{1}{2}\sigma_y^2 y^2 \frac{dt}{(dy)^2} - \frac{1}{2}\mu_y^Q y \frac{dt}{dy}. \end{cases}$$

$$(12)$$

Note that X, Y, and Z are functions of y, i.e., their values vary with position y in the state space grid (or column and row indices in AD matrix \mathbf{A}_{dt}). The adopting of different discretization schemes by different analysts in the continuous setting mirrors the adopting of different subjective state space specifications in the discrete setting. Hence the recovery consistency requirement in the continuous setting limit can be be seen in the consolidation of AD matrices (12) associated with different discretization schemes. Inconsistent recovery processes and results may also reveal in spurious arbitrage opportunities (i.e., negative prices). As a result, the consistency conditions in the continuous setting can also be formulated as a positivity requirement (to rule out arbitrages) on the limiting expression (12) of AD matrix. The next section presents this formulation, after identifying a necessary and sufficient

condition for the consistency in recovery processes.

2.3 Consistent Recovery

This section formalizes the consistency requirement in recovery results associated with different state space specifications. The necessary and sufficient condition for a consistent recovery is restrictive, indicating an elusive nature of consistent recovery processes in general.

Discrete setting

Our thought analysis addresses the recovery results obtained by two analysts using different subjective state space specifications, namely the original $S \equiv \{1, \ldots, S\}$ and the consolidated $\overline{S} \equiv \{\overline{1}, \ldots, \overline{S}\}$. For clarity, we assume that \overline{S} is netted in S. That is, given any state $\{\overline{j}\} \in \overline{S}$ in the consolidated specification, an original state $\{k\}$ either is an inclusive component of $\{\overline{j}\}$ (i.e., $\{k\} \subset \{\overline{j}\}$) or does not at all overlap with $\{\overline{j}\}$ (i.e., $\{k\} \cap \{\overline{j}\} = \emptyset$). This state space's nesting partition formalizes the single and coupled natures of states (introduced earlier), and can be characterized by a binary indicator $\mathbf{C}_{k\overline{j}}$,

$$\mathbf{C}_{k\,\overline{j}} = 1 \text{ if } \{k\} \subset \{\overline{j}\}, \qquad \mathbf{C}_{k\,\overline{j}} = 0 \text{ if } \{k\} \cap \{\overline{j}\} = \emptyset, \quad \forall \{k\} \in \mathcal{S}, \{\overline{j}\} \in \overline{\mathcal{S}}.$$
(13)

It is important to observe that these nesting partitions aim to illustrate an important feature that the consolidated specification \overline{S} is associated with unambiguously less (coarser) information structure about the state space than the original specification S. The indicators (13) together form an $S \times \overline{S}$ indicator matrix \mathbf{C} and fully characterize the mapping between the two specifications. To illustrate, for the specific Example 1, the 3 × 2 indicator matrix C mapping the original and consolidated state space specification reads

$$\{\overline{1}\} = \{1\} \\ \{\overline{2}\} = \{2,3\} \} \implies \mathbf{C} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}.$$
 (14)

The basic recovery process proceeds in two steps, i.e., constructing the implied one-period AD matrix from the observed price data (4) and then solving for its dominant eigenvalue and eigenvector (2) to recover the time and risk preferences. In the first step, the one-period AD price matrices (**A** and $\overline{\mathbf{A}}$ respectively for original \mathcal{S} and consolidated $\overline{\mathcal{S}}$ specifications) are implied from the rolling of AD assets over one more period recursively. These two recursive equation systems are versions of the same system (4), but adopted for specifications \mathcal{S} and $\overline{\mathcal{S}}$,

Original system:
$$\underbrace{\mathbf{A}_{\tau+1}}_{S \times S} = \underbrace{\mathbf{A}_{\tau}}_{S \times S} \underbrace{\mathbf{A}}_{S \times S};$$
 Consolidated system: $\underbrace{\overline{\mathbf{A}}_{\tau+1}}_{\overline{S} \times \overline{S}} = \underbrace{\overline{\mathbf{A}}_{\tau}}_{\overline{S} \times \overline{S}}, (15)$

where \mathbf{A}_{τ} and $\mathbf{A}_{\tau+1}$ contain the observed price data for S + 1 maturities $\tau \in \{1, \ldots, S + 1\}$ in the original specification (4). Similarly, $\overline{\mathbf{A}}_{\tau}$ and $\overline{\mathbf{A}}_{\tau+1}$ contain the observed price data of the consolidated AD assets for $\overline{S} + 1$ maturities $\tau \in \{1, \ldots, \overline{S} + 1\}$. Since the consolidated specification is associated with a coarser partition and information structure of the state space (as observed below (13)), it requires less data for the recovery than the original specification.

In the second step, recall from (2) that the dominant eigenvectors of \mathbf{A} and $\overline{\mathbf{A}}$ (15) are identified with the (inverse of) marginal utilities in original and consolidated specifications respectively. Therefore, relating these eigenvectors helps to relate the recovery results in different state space specifications. Consider a state $\{\overline{j}\} \in \overline{S}$ in the consolidated specification that nets several original states $\{j\} \in S$. Given the same initial current state *i*, the current AD asset prices observed by the two analysts satisfy no-arbitrage relations $\overline{A}_{\tau;i\overline{j}} = \sum_{\{j\} \subset \{\overline{j}\}} A_{\tau;ij}$, $\forall \tau \in \{1, \ldots, T\}$. Using matrix notation, assembling these no-arbitrage relations for all states $\{\overline{j}\}$ in the consolidated specification yields,⁸

$$\underbrace{\overline{\mathbf{A}}_{\tau}}_{\overline{S}\times\overline{S}} = \underbrace{\left[\mathbb{I}_{\overline{S}\times\overline{S}} \quad \mathbb{O}_{\overline{S}\times(S-\overline{S})}\right]}_{\overline{S}\times S} \underbrace{\mathbf{A}_{\tau}}_{S\times S} \underbrace{\mathbf{C}}_{S\times\overline{S}}; \qquad \qquad \underbrace{\overline{\mathbf{A}}_{\tau+1}}_{\overline{S}\times\overline{S}} = \underbrace{\left[\mathbb{I}_{\overline{S}\times\overline{S}} \quad \mathbb{O}_{\overline{S}\times(S-\overline{S})}\right]}_{\overline{S}\times S} \underbrace{\mathbf{A}_{\tau+1}}_{S\times S} \underbrace{\mathbf{C}}_{S\times\overline{S}}, \qquad (16)$$

where I denotes an identity matrix and O a matrix of zeros. Matrix equations (16) simply perform the summing of columns j's of the underlying τ -period AD price matrix \mathbf{A}_{τ} into a column $\{\overline{j}\}$ of the consolidated τ -period AD price matrix $\overline{\mathbf{A}}_{\tau}$, where $\{j\}$'s are the underlying component states of the coupled state $\{\overline{j}\}$, or $\{j\} \subset \{\overline{j}\}$. Equation (24) below illustrates this summing operation for Example 1. The no-arbitrage matrix relation (16) then transforms the recovery system (15) into,⁹

$$\overline{\mathbf{A}}_{\tau+1} = \begin{bmatrix} \mathbb{I}_{\overline{S} \times \overline{S}} & \mathbb{O}_{\overline{S} \times (S-\overline{S})} \end{bmatrix} \mathbf{A}_{\tau} \mathbf{A} \mathbf{C}.$$
(17)

Let us define a $\overline{S} \times \overline{S}$ matrix **B** that satisfies AC = CB. As a result, Equation (17) simplifies further as

$$\overline{\mathbf{A}}_{\tau+1} = \begin{bmatrix} \mathbb{I}_{\overline{S} \times \overline{S}} & \mathbb{O}_{\overline{S} \times (S-\overline{S})} \end{bmatrix} \mathbf{A}_{\tau} \mathbf{C} \mathbf{B} = \overline{\mathbf{A}}_{\tau} \mathbf{B},$$
(18)

where the second equality arises from (16). Comparing (18) with the consolidated system in (15) implies,¹⁰

$$\underbrace{\mathbf{A}}_{S \times S} \underbrace{\mathbf{C}}_{S \times \overline{S}} = \underbrace{\mathbf{C}}_{S \times \overline{S}} \underbrace{\overline{\mathbf{A}}}_{\overline{S} \times \overline{S}}.$$
(19)

Recall that while one-period AD price matrices A and \overline{A} (15) are implied from price data,

⁸That is, in matrix notation (16), these no-arbitrage relations amount to summing columns j's of the underlying τ -period AD price matrix \mathbf{A}_{τ} into a corresponding column $\{\bar{j}\}$ of the consolidated τ -period AD price matrix $\overline{\mathbf{A}}_{\tau}$, where $\{j\}$'s are the underlying component states of the coupled state $\{\bar{j}\}$, or $\{j\} \subset \{\bar{j}\}$.

⁹First, multiplying the matrix $\begin{bmatrix} \mathbb{I}_{\overline{S}\times\overline{S}} & \mathbb{O}_{\overline{S}\times(S-\overline{S})} \end{bmatrix}$ to the left, and the indicator matrix **C** to the right, of the original system in (15) produces $\begin{bmatrix} \mathbb{I}_{\overline{S}\times\overline{S}} & \mathbb{O}_{\overline{S}\times(S-\overline{S})} \end{bmatrix} \mathbf{A}_{\tau+1}\mathbf{C} = \begin{bmatrix} \mathbb{I}_{\overline{S}\times\overline{S}} & \mathbb{O}_{\overline{S}\times(S-\overline{S})} \end{bmatrix} \mathbf{A}_{\tau}\mathbf{A}\mathbf{C}$. The second equation in (16) then implies (17).

¹⁰Equations (15) and (18) show that the matrix **B** introduced above coincides with the AD price matrix $\overline{\mathbf{A}}$ in the consolidated specification. $\overline{\mathbf{A}}$ therefore also satisfies **B**'s defining identity listed below (17), $\mathbf{AC} = \mathbf{CB}$, which becomes (19).

indicator matrix \mathbf{C} (13) arises exclusively from analysts' subjective specifications. Given a set of price data driven by an underlying (objective) market model, not every (subjective) indicator matrix \mathbf{C} satisfies (19). Equivalently, not every two subjective specifications of the state space are simultaneously consistent with the observed price data. Equation (19) presents an important condition for two state space specifications to be consistent in the recovery process.

To illustrate, for the specific Example 1, the substitution of indicator matrix C (14) into Equation (19) generates the following no-arbitrage conditions

$$\begin{cases}
A_{11} = \overline{A}_{\overline{1}\overline{1}}; & A_{12} + A_{13} = \overline{A}_{\overline{1}\overline{2}} \\
A_{21} = \overline{A}_{\overline{2}\overline{1}}; & A_{22} + A_{23} = \overline{A}_{\overline{2}\overline{2}} \\
A_{31} = \overline{A}_{\overline{2}\overline{1}}; & A_{32} + A_{33} = \overline{A}_{\overline{2}\overline{2}}.
\end{cases}$$
(20)

The first two conditions are an innocuous implication of the consolidation scheme $\{1\} = \{1\}$ and $\{\overline{2}\} = \{2, 3\}$ (Figure 1). The remaining four conditions are stringent and require that $A_{21} = A_{31}$ (as both are identified with $\overline{A}_{2\overline{1}}$), and $A_{22} + A_{23} = A_{32} + A_{33}$ (as both are identified with $\overline{A}_{\overline{2}\overline{2}}$). Note that these conditions are exogenous to the price data because they arise from analysts' subjective specifications. These conditions place strong constraints on the results recovered by adopting these subjective specifications in the recovery process. The recovery consistency issue arises because these strong conditions can only be satisfied in a restrictive set of special underlying market models, being manifest in that the recovery results in different subjective specifications are incompatible, Specifically, using the pricing equation (3) for AD assets, conditions (20) only hold when the marginal utilities and transition probabilities concerning the original states $\{2\}$ and $\{3\}$ (belonging to the same consolidated state $\{\overline{2}\}$) are indistinguishable in the underlying market model (quantified in Equation (22) below). When original states $\{2\}$ and $\{3\}$ are distinguishable in the underlying model (e.g., $M_2 \neq M_3$), the recovery results in original and consolidated specifications (Figure 1) are inconsistent with each other because conditions in (20) are violated. The following necessary and sufficient condition for the recovery consistency formalizes our discussion about comparing and reconciling recovery results associated with different state space specifications.

Proposition 1 Let $S \supset \overline{S}$ denote two (subjective and netting) state space specifications of the same underlying (but unobserved) market model. The Ross's recovery results obtained in the two specifications are mutually consistent if and only if all single states $\{j\} \in S$ that correspond to a coupled state $\{\overline{j}\} \in \overline{S}$ are associated with identical characteristics (marginal utilities and transition probabilities) from the coarser specification \overline{S} 's perspective,

$$M_i = M_k, \qquad p_{i\overline{h}} = p_{k\overline{h}}; \qquad \forall \{i\}, \{k\} \subset \{\overline{j}\}; \ \{\overline{j}\}, \{h\} \in \mathcal{S}.$$

$$(21)$$

Appendix A presents a proof of this proposition. To illustrate, for the specific Example 1, the necessary and sufficient condition (21) for the recovery consistency under $S = \{1, 2, 3\}$ and $\overline{S} = \{\overline{1}, \overline{2}\}$ (with $\{\overline{1}\} = \{1\}, \{\overline{2}\} = \{2, 3\}$) contains three specific conditions

$$M_2 = M_3,$$
 $p_{21} = p_{31},$ $\underbrace{p_{22} + p_{23}}_{p_{2\overline{2}}} = \underbrace{p_{32} + p_{33}}_{p_{3\overline{2}}}.$ (22)

Qualitatively, Proposition 1 highlights an endogeneity issue inherent to the recovery framework. The state space specification is not observed prior to the recovery process yet is consequential to the recovery results. Different presumed specifications lead to possibly irreconcilable recovery results. Quantitatively, Proposition 1's necessary and sufficient condition is highly restrictive, asserting that all original states constituting a consolidated state be indistinguishable for consistent recovery results in the original and consolidated specifications. This condition can only be satisfied in a set of special underlying market models and special (compatible) subjective specifications, highlighting an elusive nature of a consistent recovery. The recovery results obtained by different analysts would be inconsistent with each other because, most likely, Proposition 1's necessary and sufficient condition is not satisfied by their subjective specifications and the underlying market model. Finally, note that when the Recovery Theorem's assumptions hold, an analyst's recovery results being inconsistent with the underlying (true) market's characteristics must be because the analyst's subjective specification violates condition (21) in its relationship with the underlying (true) specification. Being inconsistent with the true specification, the subjective one is a misspecification associated with the analyst's spurious recovery results. While in principle it could be possible to detect a misspecification, it is impractical or impossible to identify a consistent specification by ruling out virtually infinitely many misspecifications.¹¹

Continuous Setting

We recall that the recovery process in the continuous setting involves the differential equation $\mathcal{D}_{dt}^Q \mathbf{x}(y) = \log(\delta) \mathbf{x}(y)$ (8), whose numerical solution amounts to determining the eigenproblem of the associated AD price matrix \mathbf{A}_{dt} (12). In this numerical approach, the recovery consistency issue in the continuous setting then concerns a consistent construction of the AD price matrix \mathbf{A}_{dt} in different discretization (finite-difference) schemes of the underlying state space. As discussed below (12), a formulation of the consistency condition imposes an positivity requirement on \mathbf{A}_{dt} to rule out arbitrage opportunities (because their presence indicates a spurious and inconsistent recovery process).

To illustrate this consistency requirement, assume for simplicity that the state variable y_t (5) has constant risk-neutral moments $\{\mu_y^Q, \sigma_y\}$. This assumption implies that entries X, Y, Z of the AD price matrix \mathbf{A}_{dt} (12) are state-invariant. In this simple premise, the positivity requirement on these entries of \mathbf{A}_{dt} translates into the following conditions on the

¹¹Given two sets of subjective recovery results, one can check their consistency conditions. If some of these conditions are violated, then at least one of the subjective specifications is spurious. But as the true specification is unobserved, one might not know for sure which of the two subjective specifications is spurious without further testing.

finite-difference steps dt and $\frac{dy}{y}$ of time and state space,

$$\left|\frac{\sigma_y^2}{\mu_y^Q}\right| > \frac{dy}{y} > \sigma_y \sqrt{\frac{dt}{1 - r(y)dt}}.$$
(23)

Consistency conditions (23) are important to illustrate the origin of the recovery consistency issue (Section 3 below). We note that when $\{\mu_y^Q, \sigma_y\}$ are state-dependent (i.e., varying with y), the above conditions also become state-dependent. That is, the discretization $\left\{\frac{dy}{y}, dt\right\}$ of the state space and time in the recovery implementation needs to vary harmoniously with the state dynamics $\{\mu_y^Q, \sigma_y\}$ to preserve the consistency conditions (23) for the entire domain of the state variable y. That is, analysts unaware or ignorant of the state dynamics may unknowingly and subjectively adopt a discretization scheme that delivers inconsistent recovery results. Building on these observations in discrete and continuous settings, the next section elaborates on the origin of the consistency issue in recovery.

3 Origin of the Recovery Inconsistency

Being equipped with Proposition 1's necessary and sufficient characterization of a consistent recovery process, this section elucidates the origin the recovery consistency issue. We demonstrate a tradeoff between the sophistication of the state space specification (accommodating realistic recovery settings) and its effect on a non-linear error buildup in the recover results (deepening the recovery consistency issue). This tradeoff shows that the limit of the state space partition grid does not solve the recovery consistency issue. We first briefly present some basic intuitions underlying this tradeoff in both discrete and continuous settings (Section 3.1), before detailing the analysis setup (Section 3.2), results (Section 3.3) and further discussion (Section 3.4). Supporting technical derivations are relegated to Appendix B.1.

3.1 Basic Intuitions

A preliminary discussion informs our basic intuitions about the nature of the recovery consistency and a detailed quantitative analysis in subsequent sections.

In the discrete setting, the key quantity to the concept and implementation of Ross's recovery is the one-period AD asset price matrix (2), whose dimension and determination are dictated by the state space specification. When employing a subjective (but generic) specification \overline{S} of \overline{S} states, an analyst obtains the corresponding $\overline{S} \times \overline{S}$ one-period AD asset price matrix $\overline{\mathbf{A}}$. Two intuitions are in order. First, a tradeoff concerning the state space specification exists and compromises the quality of recovery results. Given a specification \overline{S} of \overline{S} states (e.g., chosen subjectively by a recovery-implementing analyst), the $\overline{S} \times \overline{S}$ AD matrix $\overline{\mathbf{A}}$ is solved from the recursive system of \overline{S} equations involving $\overline{S} + 1$ different maturities of asset price data (i.e., the second system in (15), which originates from (4)). While an increase in the number \overline{S} of states in the subjective specification improves the flexibility to approximate the underlying (to be recovered) state space more closely, it also involves a larger recovery equation system. Since the equations therein are recursive, even a small mismatch (i.e., misspecification or error) between the subjective and the underlying state space specifications¹² reverberates and amplifies non-linearly in the recovery equation system (i.e., the second system in (15)). In particular, \overline{S} also represents the non-linearity degree of the misspecification amplification and the resulting recovery error buildup (i.e. inconsistency). Note that this tradeoff among the effects of the value \overline{S} on the recovery results arise even when price data for $\overline{S}+1$ maturities is available and observed without measurement errors. The tradeoff clearly indicates that a more sophistication in the subjective state space specification (a larger \overline{S} , and error-free associated price data) do not solve the recovery consistency issue.¹³

 $^{^{12}}$ This small mismatch results from a better approximation associated with a larger number \overline{S} of states in the subjective specification.

¹³This tradeoff is an alternative and intuitive way to paraphrase the Proposition 1's result that the recovery process is consistent if and only if restrictive conditions specified in that proposition hold.

Second, an inadvertent (and irreversible) loss of information concerning the recovery implementation exists and skews the recovery results obtained by the analyst. Given that the analyst's subjective specification $\overline{\mathcal{S}}$ may differ from the underlying (and unobserved) specification \mathcal{S} , a consolidation of the underlying τ -period AD asset prices into those in the subjective specification is needed, $\overline{A}_{\tau;i\bar{j}} = \sum_{\{j\} \subset \{\bar{j}\}} A_{\tau;ij}, \forall \tau \in \{1, \ldots, T\}, \{i\} \in \mathcal{S}, \{\bar{j}\} \in \overline{\mathcal{S}}$ (as discussed above Equation (16)). Evidently, many different possible sets of τ -period AD prices $\{A_{\tau;ij}\}$ can generate the same configuration of consolidated τ -period AD prices $\{\overline{A}_{\tau;i\bar{j}}\}$ perceived by the analyst, i.e., a many-to-one ambiguity between the underlying recoveries and the subjective recovery. In other words, a same set of recovery results obtained by the analyst (using price data $\{\overline{A}_{\tau;i\bar{j}}\}$) may correspond to different underlying market models (each associated with a possible set of price data $\{A_{\tau;ij}\}$). When these different market models conflict with each other, the many-to-one ambiguity indicates that the recovery results obtained in the subjective specification, albeit unique, do not necessarily distinguish and implicate the true underlying market model, indicating a recovery consistency issue.¹⁴ Intuitively, this ambiguity arises from the fact that the subjective specification $\overline{\mathcal{S}}$ features a coarser partition and information structure of the state space than the underlying \mathcal{S} as observed below (13). The construction of τ -period AD prices $\{\overline{A}_{\tau;i\bar{j}}\}$ by the analyst for the recovery then inadvertently (and irreversibly) induces a loss of information.

To illustrate, for the specific Example 1, assume the current underlying state is {1}. We consider two (hypothetical) underlying τ -period AD price matrices \mathbf{A}_{τ}^{I} and \mathbf{A}_{τ}^{II} listed in (24) that are associated with two different underlying market models. The consolidation (16), which amounts to summing certain columns of the underlying τ -period AD price matrices (per Footnote 8 and using indicator matrix \mathbf{C} (13)), yields identical consolidated τ -period

¹⁴Even when more price data are available (than required by a just-identified recovery system (4)) and employed in the best-fit recovery approach, the loss of information and recovery consistency issue remain (Appendix C.2).

AD price matrices $\overline{\mathbf{A}}_{\tau}^{I} = \overline{\mathbf{A}}_{\tau}^{II}$,

$$\mathbf{A}_{\tau}^{I} = \begin{bmatrix} 1 & a & b \\ 2 & c & d \\ 3 & e & f \end{bmatrix}, \quad \mathbf{A}_{\tau}^{II} = \begin{bmatrix} 1 & \frac{a+b}{2} & \frac{a+b}{2} \\ 2 & \frac{c+d}{2} & \frac{c+d}{2} \\ 3 & \frac{e+f}{2} & \frac{e+f}{2} \end{bmatrix} \implies \overline{\mathbf{A}}_{\tau}^{I} = \overline{\mathbf{A}}_{\tau}^{II} = \begin{bmatrix} 1 & a+b \\ 2 & c+d \end{bmatrix}. \quad (24)$$

That is, information about the specific underlying market model (I or II) and its characteristics (time and risk preferences and transition probabilities) is lost at the analyst's recovery level, which is built on the asset prices in indistinguishable matrices $\overline{\mathbf{A}}_{\tau}^{I} = \overline{\mathbf{A}}_{\tau}^{II}$.

In the continuous setting, the AD matrix is replaced by the risk-neutral infinitesimal operator \mathcal{D}^Q (11), whose finite-difference representation (12) discretizes the recovery differential equation $\mathcal{D}^{Q}\mathbf{x}(y) = -\rho\mathbf{x}(y)$ (8) for a numerical recovery solution. In this infinitesimal limit of the state space grid partition, the violation of the consistency conditions (23) implies arbitrage opportunities (i.e., negative AD asset prices), and hence, an inconsistent recovery process. Because these conditions are governed by the underlying state variable's dynamics $\{\mu_y^Q,\sigma_y\}$ they are violated when the analyst adopts a discretization scheme incompatible to the underlying dynamics. Specifically, given a state (relative) volatility $\left|\frac{\sigma_y}{\mu_x}\right|$, a sufficiently fine state space grid (sufficiently small dy) and sufficiently frequent sampling of asset maturities (sufficiently small dt) are needed. Otherwise, it is possible that $\left|\frac{\sigma_y}{\mu_y^Q}\right| < \frac{1}{\sigma_y} \frac{dy}{y}$ or $\left|\frac{\sigma_y}{\mu_y^Q}\right| < \sqrt{\frac{dt}{1-r(y)dt}}$, which violate conditions (23) and yield inconsistent recovery results for the analyst. A harmonious time and state space discretization scheme $\left\{ dt, \frac{dy}{y} \right\}$ required in the recovery process reflects an intuitive notion that one needs a sufficiently fine sampling of data to probe a sufficiently fast-moving (volatile) underlying state dynamics. It is important to observe that the limit of high state space partition resolution, $dy \to 0$, does not assure the recovery consistency if the maturity sampling is not commensurately frequent (i.e., when dtdoes not commensurately shrink, $\frac{dy}{y} < \sigma_y \sqrt{\frac{dt}{1-r(y)dt}}$, violating (23)). As this standard limit corresponds to a discrete setting with infinitely many states, the observation shows that the sophistication of the state space specification alone does not solve the recovery consistency issue. Sections 3.2-3.4 substantiate this observation with a quantitative setup and in-depth analysis.

In retrospective, Proposition 1's restrictive condition is needed to assure that underlying component states (belonging to a state in the consolidated specification) are indistinguishable. Such a restrictive structure prevents a loss of information on the underlying unobserved state space S when the analyst employs a subjective specification $\overline{S} \neq S$, hence preserving the recovery consistency.

3.2 Perturbative Setup

To examine how recovery results vary with the sophistication of the state space specification, we adopt a thought-experiment setup with two subjective specifications with distinct sophistication level for the same underlying state space. By varying the difference between the two specifications, our setup's configuration explicitly demonstrates the origin and persistence of the recovery consistency issue as the two specifications converge.

Assume that the underlying state space $S = \{1, 2, 3, 4\}$ is composed of S = 4 original states. Whereas, the first analyst's subjective state space specification (indexed by superscript *a*) has $\overline{S}^a = 2$ states, the second analyst's specification (indexed by *b*) has $\overline{S}^b = 3$ states. Let the analysts' consolidation schemes be as follows (also depicted in Figure 2)

$$\frac{\text{(Simple) Specification }a:}{\overline{\mathcal{S}}^{a} = \{\overline{1}^{a}, \overline{2}^{a}\}, \text{ with:}} \begin{cases} \frac{\text{(Sophisticated) Specification }b:}{\overline{\mathcal{S}}^{b} = \{\overline{1}^{b}, \overline{2}^{b}, \overline{3}^{b}\}, \text{ with:}} \\ \overline{\mathcal{S}}^{b} = \{\overline{1}^{b}, \overline{2}^{b}, \overline{3}^{b}\}, \text{ with:} \end{cases} \\ \{\overline{1}^{a}\} \equiv \{1\}, \ \{\overline{2}^{a}\} \equiv \{2, 3, 4\}, \end{cases} \begin{cases} \overline{1}^{b}\} \equiv \{1\}, \ \{\overline{2}^{b}\} \equiv \{2\}, \ \{\overline{3}^{b}\} \equiv \{3, 4\}. \end{cases} \end{cases}$$

Let the current state be the single state $\{1\} = \{\overline{1}^a\} = \{\overline{1}^b\}$. Our thought-experiment analysis starts with a marginal utility configuration for the underlying state space characterized by



Figure 2: Underlying (true, but unobserved) and subjective state space specifications for different analysts a and b associated with the consolidation scheme (25). The subjective specification of analyst a is more sophisticated than that of analyst b.

a small real parameter ε as follows,

$$M_1 = 1, \ M_2 = M, \ M_3 = M + 2\varepsilon, \ M_1 = M + 3\varepsilon, \ \text{with } |\varepsilon| \ll 1.$$
 (26)

Several rationales motivate this configuration. First, since marginal utilities are determined up to a multiplicative constant, without loss of generality we normalize the marginal utility in the first state to one. Second, when $\varepsilon = 0$, marginal utilities are identical in the three original states {2,3,4} (26), satisfying the requirement on marginal utilities for the recovery consistency (Proposition 1). The thought experiment depicted in Figure 2 also employs underlying transition probability inputs that satisfy Proposition 1's requirement on the transition probabilities. As a result, the recovery results associated with a's, b's, and the underlying specifications are mutually consistent when $\varepsilon = 0$. Accordingly, we refer to the marginal utility configuration (26) associated with the specific value $\varepsilon = 0$ as the consistent configuration, and ε as the deviation parameter from the consistent configuration.¹⁵ Third, analyst b's specification ($\overline{S}^b = 3$) is closer to the underlying specification

¹⁵ Given a value $\varepsilon \neq 0$, the configuration (26) presents the "true" (underlying but unobserved) marginal utilities to be recovered. For an analytical exposition, we consider a sufficiently small value ε so that the "true" underlying configuration is close to the consistent configuration (associated with $\varepsilon = 0$ in (26) and satisfying Proposition 1's conditions). That is, the special value $\varepsilon = 0$ does not represent the "true" configuration in the thought experiment. Instead, $\varepsilon = 0$ represents the unperturbed configuration, i.e., the leading component of the setting in which ε is non-zero and small.

(S = 4) than *a*'s $(\overline{S}^a = 2)$. Furthermore, as the spread between the underlying marginal utilities $(\{M_2 - M_1, M_3 - M_2, M_4 - M_3\})$ is not equally distributed, analyst *b*'s coupled state $\{\overline{3}^b\} \equiv \{3, 4\}$ is more homogeneous than *a*'s coupled state $\{\overline{2}^a\} \equiv \{2, 3, 4\}$. These features aim to assure that *b*'s subjective specification is more sophisticated than *a*'s¹⁶ and set the stage for our comparative analysis in the thought experiment.

Two comparative statics of interest

A priori there are two comparative statics concerning the recovery consistency issue.

- C1. $\varepsilon \to 0$ while fixing the state space specifications: This comparative statics pertains to variations associated with a vanishing deviation from the consistent configuration. Because the underlying configuration (26) coincides with the consistent configuration at $\varepsilon = 0$, this comparative statics helps to reveal how analysts' recovery results approach the underlying market's risk and time preferences as $\varepsilon \to 0$.
- C2. $\overline{S}^a \to \overline{S}^b$ while fixing $\varepsilon \neq 0$: This comparative statics pertains to variations from a subjective specification \overline{S}^a to another more sophisticated \overline{S}^b . It helps to reveal whether getting closer to the underlying specification improves the recovery results when the underlying specification S does not belong to the restrictive premise of Proposition 1 (i.e., $\varepsilon \neq 0$).

The first comparative statics C1 leads to standard convergence results. Because $\varepsilon = 0$ corresponds precisely to the consistent configuration of Proposition 1, standard regularity conditions imply that in the limit of vanishing deviation, $\varepsilon \to 0$, the recovery results obtained by both analysts converge to the underlying, i.e., the recovery consistency issue disappears in this limit (see explicit solutions (30) below).

In contrast, the second comparative statics C2 leads to surprising results. It shows that

¹⁶These assumptions are innocuous. Our perturbative analysis below enables closed-form solutions to any state space configurations as long as ε is sufficiently small.

agent *a*'s primitive specification is capable of delivering superior recovery results (compared to agent *b*'s sophisticated specification). Intuitively, non-linear effect of the specification on the recovery process generates a subtle pattern that the recovery consistency does not necessarily improve with the sophistication of the subjective specification. The analysis below derives this comparative statics and elucidates how the non-linearity stemming from the sophistication of the state space specification ends up impairing the consistency of the recovery results in that specification (Equation (35) and the related discussion below).

3.3 Perturbative Results

We briefly list the main steps and quantitative results of the perturbative approach, before presenting a detailed analysis. Technical derivations are in Appendix B.1. Assuming standard regularity conditions, a generic recovery quantity $Q^i(\varepsilon)$ in the specification $i \in \{a, b\}$ has the following perturbative expansion in the leading order

$$Q^{i}(\varepsilon) = \underbrace{Q}_{\text{unperturbed component}} + \underbrace{\varepsilon P^{i}}_{\text{perturbative component}}, \quad i \in \{a, b\},$$
(27)

where the unperturbed quantity Q is associated with $\varepsilon = 0$ (or the consistent configuration). Per the discussion earlier (Footnote 15), the unperturbed Q is not the "true" underlying quantity, but the zero-order term in the perturbative expansion.¹⁷ In the leading order, $Q^i(\varepsilon)$ deviates from the unperturbed Q by an amount commensurate (linear) with ε , where $Q^i(\varepsilon)$, Q, and P^i in (27) are quantities (scalars, vectors, or matrices) of same dimensions and order of magnitudes.

Adopting the recovery process in the thought-experiment setting (Sections 2.2 and 2.3) and the perturbative expansion (27) for a small ε , the recursive recovery system (15) that solves the consolidated AD price matrix $\overline{\mathbf{A}}^{i}$ in specification $i \in \{a, b\}$, has the following

¹⁷In fact, only when $\varepsilon = 0$ represents the "true" underlying configuration, then Q is the "true" underlying quantity.

form,¹⁸

$$\underbrace{\overline{\mathbf{A}}_{\tau+1}^{i}(\varepsilon)}_{\overline{S}^{i}\times\overline{S}^{i}} = \underbrace{\overline{\mathbf{A}}_{\tau}^{i}(\varepsilon)}_{\overline{S}^{i}\times\overline{S}^{i}}, \underbrace{\overline{\mathbf{A}}^{i}(\varepsilon)}_{\overline{S}^{i}\times\overline{S}^{i}}, \text{ with } \begin{cases} \overline{\mathbf{A}}_{\tau}^{i}(\varepsilon) &= \overline{\mathbf{A}}_{\tau}^{i} + \varepsilon \overline{\mathbf{B}}_{\tau}^{i}, \\ \overline{\mathbf{A}}_{\tau+1}^{i}(\varepsilon) &= \overline{\mathbf{A}}_{\tau+1}^{i} + \varepsilon \overline{\mathbf{B}}_{\tau+1}^{i}, \quad i \in \{a, b\}, \\ \overline{\mathbf{A}}^{i}(\varepsilon) &= \overline{\mathbf{A}}^{i} + \varepsilon \overline{\mathbf{B}}^{i}, \end{cases}$$
(28)

where matrix $\overline{\mathbf{A}}_{\tau}^{i}(\varepsilon)$ contains the AD asset prices initiated on the current state and maturing in $\tau \in \{1, \ldots, \overline{S}^{i} + 1\}$ periods associated with specification \overline{S}^{i} . These current AD asset prices are observable, arising explicitly from consolidating the underlying S into \overline{S}^{i} . By the same reason, matrices $\overline{\mathbf{A}}_{\tau+1}^{i}(\varepsilon)$, $\overline{\mathbf{A}}_{\tau}^{i}$, $\overline{\mathbf{B}}_{\tau}^{i}$, $\overline{\mathbf{A}}_{\tau+1}^{i}$, $\overline{\mathbf{B}}_{\tau+1}^{i}$, which concern only the AD assets initiated on the current state, are also observable. In contrast, one-period AD price matrix $\overline{\mathbf{A}}^{i}(\varepsilon)$ (and its components $\overline{\mathbf{A}}^{i}$, $\overline{\mathbf{B}}^{i}$ (28)), which concern the AD assets initiated on all possible states in S^{i} , are implied as follows. Substituting the expansions of $\overline{\mathbf{A}}_{\tau}^{i}(\varepsilon)$ and $\overline{\mathbf{A}}^{i}(\varepsilon)$ into the recursive recovery system (all in (28)) and matching terms of the same order in ε imply the perturbative component $\overline{\mathbf{B}}^{i}$, and hence, the entire AD price matrix $\overline{\mathbf{A}}^{i}(\varepsilon)$ in the leading order,

$$\overline{\mathbf{B}}^{i} = \left(\overline{\mathbf{A}}_{\tau}^{i}\right)^{-1} \left(\overline{\mathbf{B}}_{\tau+1}^{i} - \overline{\mathbf{B}}_{\tau}^{i}\overline{\mathbf{A}}^{i}\right), \qquad \overline{\mathbf{A}}^{i}(\varepsilon) = \overline{\mathbf{A}}^{i} + \varepsilon \overline{\mathbf{B}}^{i} = \overline{\mathbf{A}}^{i} + \varepsilon \left(\overline{\mathbf{A}}_{\tau}^{i}\right)^{-1} \left(\overline{\mathbf{B}}_{\tau+1}^{i} - \overline{\mathbf{B}}_{\tau}^{i}\overline{\mathbf{A}}^{i}\right).$$
(29)

These matrix equations help to demonstrate how the subjective specification S^i affects the implied AD price matrix $\overline{\mathbf{A}}^i(\varepsilon)$, from which analyst *i*'s recovery results arise. We note that matrices in (29) are related to the Vandermonde matrix, and therefore, have explicit solutions.¹⁹ Specifically, we start with the perturbative expansions of the dominant eigenvalue

¹⁸Specifically, the recovery process in the thought experiment for a subjective specification $\overline{\mathcal{S}}^i$ consists of three steps to determine (i) the (observable) matrix $\overline{\mathbf{A}}^i_{\tau}$ of current AD prices maturing in different periods $\tau \in \{1, \ldots, S^i + 1\}$ (by consolidating the current AD prices in the underlying specification (16)-(17)), (ii) the (implied) AD price matrix $\overline{\mathbf{A}}^i$ (by solving the recursive recovery equation system in the subjective specification $\overline{\mathbf{A}}^i_{\tau+1} = \overline{\mathbf{A}}^i_{\tau} \overline{\mathbf{A}}^i$ (15)), and (iii) the (implied) time discount factors and marginal utilities (by solving for the dominant eigenvalue and eigenvector of the AD price matrix $\overline{\mathbf{A}}^i = \overline{\delta}^i \overline{\mathbf{x}}^i$ in the subjective specification (2)).

¹⁹We discuss the relation and relevance of the Vandermonde matrix in the recovery setting in Appendix

 $\overline{\delta}_1^i(\varepsilon)$ and the associated (right) eigenvector $\overline{\mathbf{x}}_1^i(\varepsilon)$ of AD price matrix $\overline{\mathbf{A}}_i(\varepsilon)$,

$$\overline{\mathbf{A}}^{i}(\varepsilon)\overline{\mathbf{x}}_{1}^{i}(\varepsilon) = \overline{\delta}_{1}^{i}(\varepsilon)\overline{\mathbf{x}}_{1}^{i}(\varepsilon), \quad \text{with} \quad \begin{cases} \overline{\delta}_{1}^{i}(\varepsilon) &= \delta_{1} + \varepsilon \Delta \delta^{i}, \\ \overline{\mathbf{x}}_{1}^{i}(\varepsilon) &= \overline{\mathbf{x}}_{1}^{i} + \varepsilon \Delta \overline{\mathbf{x}}^{i}, \end{cases} \quad i \in \{a, b\}.$$
(30)

Note that the unperturbed dominant eigenvalue is the same for both analysts, $\delta_1^a = \delta_1^b = \delta_1$, because $\varepsilon = 0$ corresponds to the consistent configuration, in which their recovered time discount factors are identical (Proposition 1 and Footnote 15). Substituting the perturbative expansions into the eigenequation (the first equation in (30)) and matching terms of the same order in ε yield the perturbative components, and hence, entire leading-order expressions of dominant eigenvalue $\overline{\delta}_1^i(\varepsilon)$ and eigenvector $\overline{\mathbf{x}}_1^i(\varepsilon)$ under analyst *i*

$$\overline{\delta}_{1}^{i}(\varepsilon) = \delta_{1} + \varepsilon \,\Delta \delta^{i} = \delta_{1} + \varepsilon \,\frac{(\overline{\mathbf{w}}_{1}^{i})'\overline{\mathbf{B}}^{i}\overline{\mathbf{x}}_{1}^{i}}{(\overline{\mathbf{w}}_{1}^{i})'\overline{\mathbf{x}}_{1}^{i}},$$

$$\overline{\mathbf{x}}_{1}^{i}(\varepsilon) = \overline{\mathbf{x}}_{1}^{i} + \varepsilon \,\Delta \overline{\mathbf{x}}^{i} = \overline{\mathbf{x}}_{1}^{i} + \varepsilon \,\sum_{k=2}^{\overline{S}^{i}} \frac{(\overline{\mathbf{w}}_{k}^{i})'\overline{\mathbf{B}}^{i}\overline{\mathbf{x}}_{1}^{i}}{(\delta_{1} - \delta_{k}^{i})(\overline{\mathbf{w}}_{k}^{i})'\overline{\mathbf{x}}_{k}^{i}}\overline{\mathbf{x}}_{k}^{i},$$
(31)

where $\overline{\mathbf{x}}_{k}^{i}$ and $\overline{\mathbf{w}}_{k}^{i}$ denote respectively the k-th right and left eigenvectors of the $\overline{S}^{i} \times \overline{S}^{i}$ unperturbed AD price matrix $\overline{\mathbf{A}}^{i}$. Solution (31) shows that, already in the leading (linear) order of ε , all unperturbed eigenvectors are tangled with each other in their contributions to the dominant eigenvector $\overline{\mathbf{x}}_{1}^{i}(\varepsilon)$ and the associated recovered marginal utilities. Intuitively, an eigenvector $\overline{\mathbf{x}}_{k}^{i}$ closer to the dominant one $\overline{\mathbf{x}}_{1}^{i}$ (i.e., a smaller $(\delta_{1} - \delta_{k}^{i})$) is tangled more with $\overline{\mathbf{x}}_{1}^{i}$ (i.e., a larger $\frac{1}{\delta_{1}-\delta_{k}^{i}}$) and hence has a stronger impact on the recovered (perturbed) eigenvector $\overline{\mathbf{x}}_{1}^{i}(\varepsilon)$. Relating matrix equations (29) to Vandermonde matrix enables an analytical expression for $\overline{\mathbf{B}}^{i}$, and hence, for the recovery results for both analysts $i \in \{a, b\}$ (Equations (78)-(77), Appendix B.1). In particular, the time discount factors recovered by B.1, see Equation (81). the two analysts are

$$\delta_1^a(\varepsilon) = \delta_1 + \varepsilon \times \frac{\delta_1 \overline{x}_{12}^a}{\overline{x}_{11}^a} \times \frac{\delta_1 (P_3 - P_2) - \delta_2^a (P_2 - P_1)}{\delta_1 - \delta_2^a},\tag{32}$$

$$\delta_1^b(\varepsilon) = \delta_1 + \varepsilon \times \frac{\delta_1 \overline{x}_{13}^b}{\overline{x}_{11}^b} \times \frac{\delta_2^b \delta_3^b (P_2 - P_1) - \delta_1 (\delta_2^b + \delta_3^b) (P_3 - P_2) + \delta_1^2 (P_4 - P_3)}{(\delta_1 - \delta_2^b) (\delta_1 - \delta_3^b)}, \tag{33}$$

where $\overline{\mathbf{x}}_{1k}^i$ is the k-th element of the dominant right eigenvector $\overline{\mathbf{x}}_1^i$ of the one-period AD price matrix $\overline{\mathbf{A}}^i$, which is also the (unperturbed) inverse marginal utility in state $\{k\}$ (see Equation (34) below). P_{τ} concerns the transition probabilities (in the underlying specification S) from current state $\{1\}$ to states $\{3,4\}$ in τ periods; $P_{\tau} \equiv 2p_{t,t+\tau}(1,3) + 3p_{t,t+\tau}(1,4)$, where $\tau \in \{2,3,4\}$. The perturbative recovery results are derived in Appendix B.1 (Equations (79), (80)) Similar analytical perturbative expressions are obtained for the dominant eigenvectors (or, recovered marginal utilities) $\overline{\mathbf{x}}^i(\varepsilon)$, $i \in \{a, b\}$ by substituting the explicit expression for $(\overline{\mathbf{w}}_k^i)' \overline{\mathbf{B}}^i \overline{\mathbf{x}}_1^i$ (79) into Equation (31).

3.4 Perturbative Analysis

The expressions derived above set the stage for an analysis on the relationship between the subjective (input) specification and the associated recovery results, addressing the two comparative statics C1 (on $\varepsilon \to 0$) and C2 (on \overline{S}^a vs. \overline{S}^b) listed in Section 3.2. Concerning the first comparative statics C1, the limit of a vanish deviation from the consistent configuration is straightforward. As $\varepsilon \to 0$, both analysts' time discount factors $\overline{\delta}^a(\varepsilon)$ (32) and $\overline{\delta}^b(\varepsilon)$ (33) converge to the underlying δ_1 . In the same limit, their recovered dominant eigenvectors (31) converge respectively to \mathbf{x}_1^a and \mathbf{x}_1^b . As $\varepsilon = 0$, (26) shows that these eigenvectors belong to the consistent setting of Proposition 1,

$$\overline{\mathbf{x}}_{1}^{a} = \left(\frac{1}{M_{1}}, \frac{1}{M_{2}}\right)', \qquad \overline{\mathbf{x}}_{1}^{b} = \left(\frac{1}{M_{1}}, \frac{1}{M_{2}}, \frac{1}{M_{2}}\right)', \qquad (34)$$

where M_i 's denote the underlying marginal utilities (26). Therefore, in the limit $\varepsilon \to 0$ of the comparative statics C1, both analysts' recovered risk preferences also converge to the underlying.

Concerning the second comparative statics C2, the differential effect of subjective specifications \overline{S}^a and \overline{S}^b on the recovery results is subtle. While \overline{S}^b is closer (more sophisticated) to the underlying S than \overline{S}^a , the sophistication does not translate into a superior (closer to the underlying) recovery results for analyst b. This tradeoff arises because agent b's more sophisticated subjective specification accommodates a higher number of states ($\overline{S}^b > \overline{S}^a$), requires more maturities of the price data input, and relies on a recovery equation system of higher dimension and degree. Misspecification and approximation error between the subjective and underlying specifications build up in the recovery process, facilitating the recovery consistency issue. The perturbative formalism demonstrates this adverse effect and the tradeoff.

First, the consistency issue originates in the recursive recovery system (29) that solves the implied AD matrix $\overline{\mathbf{A}}_i(\varepsilon)$. As the subjective specification and consolidation are instrumental to the recovery consistency issue underlying Proposition 1, their presence in the construction of AD asset prices in $\overline{\mathbf{A}}_{\tau}^i$ and $\overline{\mathbf{B}}_{\tau}^i$ (29) signifies that they remain instrumental for the recovery consistency in the perturbative setting. Intuitively, any errors in observing these AD asset prices (due to specification, approximation, and consolidation) have a non-linear effect on the implied AD matrix $\overline{\mathbf{A}}_i(\varepsilon)$ (due to the matrix inversion $(\overline{\mathbf{A}}_{\tau}^i)^{-1}$ in (29)). The more sophisticated the subjective specification \overline{S}^i is, the more asset price maturities $\overline{S}^i + 1$ are needed, the higher dimension $\overline{S}^i \times \overline{S}^i$ the to-be-inverted matrix $\overline{\mathbf{A}}_{\tau}^i$ has, and the more tangled and non-linear the effect of the subjective misspecification \overline{S}^i on the implied AD matrix $\overline{\mathbf{A}}_i(\varepsilon)$ is. Hence, matrix solutions (29) hint at an adverse impact of the sophistication of specification on the implied AD matrix.

Second, the consistency issue is passed on to and reflected in the recovery results obtained in the eigenproblem of the implied AD matrix. The dominant eigenvalues $\overline{\delta}_1^a(\varepsilon)$ (32) and $\overline{\delta}_{1}^{b}(\varepsilon)$ (33), or the time discount factors recovered by the two analysts *a* and *b*, exhibit the specification's non-linear effect on the recovery. These expressions clearly show that the time discount factor recovered by the sophisticated analyst *b* is more non-linear than that by the primitive analyst *a*. This non-linearity stems from the inversion of a higher-dimensional $\overline{S}^{b} \times \overline{S}^{b}$ matrix \overline{A}_{τ}^{b} mentioned above. Since the unperturbed dominant eigenvectors are consistent for the two analysts (34), we have $\overline{x}_{11}^{a} = \overline{x}_{11}^{b} = \frac{1}{M_1}$, and $\overline{x}_{12}^{a} = \overline{x}_{13}^{b} = \frac{1}{M_2}$, and the difference between the two recovered time discount factors (32), (33) boils down to the following perturbative factors

$$\frac{\delta_1(P_3 - P_2) - \delta_2^a(P_2 - P_1)}{\delta_1 - \delta_2^a} \quad \text{vs.} \quad \frac{\delta_2^b \delta_3^b(P_2 - P_1) - \delta_1(\delta_2^b + \delta_3^b)(P_3 - P_2) + \delta_1^2(P_4 - P_3)}{(\delta_1 - \delta_2^b)(\delta_1 - \delta_3^b)}.$$
(35)

Observe that \overline{S}^a and \overline{S}^b originate from the same underlying specification S, and Proposition 1's consistency conditions bind their (unperturbed) dominant eigenmodes ($\delta_1^a = \delta_1^b = \delta_1$, $\overline{x}_{11}^a = \overline{x}_{11}^b$, and $\overline{x}_{12}^a = \overline{x}_{13}^b$ (34)). The higher-order eigenmodes associated with \overline{S}^a and \overline{S}^b , however, are largely exogenous to each other.²⁰ As a result, when AD price matrix \overline{A}^b has a more clustered spectrum than \overline{A}^a , analyst b's perturbative factor can be larger than that of analyst a by an order of magnitude (35). As a result, recovery results in specification \overline{S}^b can deviate from the underlying substantially more than those in specification \overline{S}^a . The more sophisticated a subjective specification is, the more non-linear are its perturbative factors (35), opening the possibility for larger recovery inconsistencies. In this regard, note that the derivation underlying the perturbative analysis holds for general state space specifications with any finite number of states (via to a connection to the Vandermonde matrix, Appendix B.1).

Alternatively, the recovery consistency issue can also be seen in the continuous setting, in

²⁰To see this in the thought-experiment setting, note that given an entire characteristic set of marginal utilities, time discount factor, and transition probabilities in a subjective specification $\overline{\mathcal{S}}^i$, we cannot deduce the corresponding characteristic set for the underlying \mathcal{S} , and therefore, the characteristic set for another subjective specification $\overline{\mathcal{S}}^k$. This is because the information about characteristics in the underlying \mathcal{S} is (partially but irreversibly) lost in the subjective specification (see (24) and the associated discussion).
which the AD price matrix \mathbf{A}_{dt} (12) for an infinitesimal time horizon dt is constructed by discretizing the underlying continuous state dynamics. The discretization scheme is subjective, i.e., two different analysts may choose different sets $\left\{ dt, \frac{dy}{y} \right\}$ of time and state space steps for the discretization, resulting in different subjective elements $\{X, Y, Z\}$ for the matrix \mathbf{A}_{dt} . A subjective and inharmonious choice of $\left\{ dt, \frac{dy}{y} \right\}$ can make one or more elements in $\{X, Y, Z\}$ (12) negative, implying arbitrage opportunities and recovery inconsistencies. In particular, a more sophisticated state space discretization scheme (i.e., small $\frac{dy}{y}$) requires a fine time discretization (i.e., commensurately small dt) to assure that $X > 0.^{21}$ Intuitively, probing the state space with a high resolution (i.e., small $\frac{dy}{y}$) requires sufficiently high-frequency data (sufficiently small dt). Moreover, a harmonious choice of $\left\{dt, \frac{dy}{y}\right\}$ that respects the positivity of $\{X, Y, Z\}$ also depends on the underlying state dynamics (Footnote 21). That is, more sophisticated subjective state space specifications do not necessarily improve the recovery results and their consistencies, specially when these specifications are selected without the knowledge of the underlying state space structure. This observation is in line with the findings above on the relationship between subjective specifications and recovery results in discrete setting.

4 Recovery Inconsistency Direction

Building upon the analysis above on the origin of the recovery consistency issue, this section examines the direction of consistencies, i.e., under what economic premises the recovery results associated with a subjective state space specification are above (overshooting) or below (undershooting) the market's underlying risk and time preferences. To study the recovery overshooting and undershooting, we employ a consumption setting calibrated to the U.S. economy (Section 4.1) and discuss the relevant economic features, such as aspects

²¹ Recall from (12) that $X \equiv 1 - r(y)dt - \sigma_y^2 y^2 \frac{dt}{(dy)^2}$. To assure that X > 0, we need $dt < \left[r(y) + \sigma_y^2 \frac{y^2}{(dy)^2}\right]^{-1}$. Evidently, a small $\frac{dy}{y}$ generates a small upper bound on dt. Note that this upper bound also depends on the underlying parameters σ_y^2 and r(y).

of a rare adverse event in the state space distribution, that determine the direction of the recovery inconsistency (Section 4.2). Details on the calibration procedure and supporting technical derivations are relegated to Appendix B.2.

4.1 Calibration and Consolidation Setup

Our analysis of the recovery inconsistency direction again adopts a tractable thought-experiment setup in which the underlying three-state model S is calibrated to stylize the U.S. consumption and aggregate stock market dynamics. Model S's asset prices (risk-free rate and stock index return) are obtained from the consumption dynamics in a standard setting of endowment economy and representative agent with a constant relative risk aversion (CRRA). Since the calibrated (underlying) market model S is unobserved in the thought experiment, an analyst attempts to recover the underlying model using a subjective two-state specification \overline{S} The consolidation of the analyst's subjective specification \overline{S} and recovery results with the underlying model S's then reveals the recovery inconsistency direction and the responsible economic features.

Calibration: For the calibration of the underlying model S, we use as input data the U.S. household real consumption expenditure, S&P 500 index level, and 3-month Treasury Bill rate from 1985Q1 to 2022Q4 and at quarterly frequency. Our calibration employs the generalized method of moments (GMM) procedure to match five model-implied stationary moments with their counterparts in the data, namely, the expected consumption growth μ_c , the volatility of consumption growth σ_c , the risk-free rate r_{t+1}^f , the S&P 500 risk premium rx_t^s , and the volatility of S&P 500 return σ_s . Table 1 presents the summary (non-annualized) statistics of the consumption and asset price inputs employed in our calibration. To match the basic consumption characteristics, we calibrate three stylized underlying states $S = \{1, 2, 3\}$, in which $\{1\}$ denotes the disaster, $\{2\}$ the normal, and $\{3\}$ the boom state of the consumption. Since the U.S. consumption growth distribution concentrates pronouncedly around the mean, we employ respectively 0.1th and 99th percentiles of the time-series consumption growth to

Variable	Mean	Std Dev	Min	25th	Median	75th	Max
Consumption growth	0.0068	0.0132	-0.0990	0.0037	0.0068	0.010	0.1038
Risk-free rate	0.0076	0.0063	-0.00003	0.0006	0.0073	0.0127	0.0223
Stock index return	0.0232	0.0676	-0.2356	-0.0132	0.0307	0.0642	0.1909
Risk premium	0.0156	0.0674	-0.2379	-0.0197	0.0203	0.0587	0.1782

Table 1: Summary statistics of the consumption and asset price inputs to the calibration (reported moments are not annualized). Risk premium denotes the return of the stock index in excess of the risk-free rate (i.e., excess return).

delineate the disaster and boom states, and the average consumption growth to identify the normal state. To match the basic asset price characteristics, we embed the three-state space S in a temporal setting of one period and two dates $\{t, t + 1\}$. We employ a consumptionbased asset pricing model (CCAPM) featuring a representative agent with CRRA γ to price and calibrate the risk-free bond and the stock market (as a contingent claim on the aggregate consumption). Fixing the underlying risk and time preferences to be $\gamma = 15$ (the relative risk aversion) and $\delta = 0.98$ (the time discount factor) for simplicity, the asset price calibration determines the underlying 3×3 one-period transition probability matrix **P** of the temporal setting (Appendix B.3 contains further details on the calibration procedure, GMM moment conditions and estimation). Table 2 presents the calibration results for the annualized moments of the consumption and asset prices. Note that the stock index return volatility is

Description	Model	Data	
Mean consumption growth	0.0257	0.0273	
Consumption growth volatility	0.0467	0.0264	
Mean risk-free rate	0.0269	0.0303	
Mean risk premium (stock)	0.0688	0.0625	
Stock index return volatility	0.0487	0.1351	

Table 2: Calibration results of the consumption and asset price (annualized) moments. Risk premium denotes the return of the stock index in excess of the risk-free rate (i.e., excess return).

significantly lower, while the consumption growth volatility is higher, in the model than in the data. These numerical values result from the constraint to match the stock index risk premium in the calibration and data, reflecting the well-known equity risk premium puzzle for the current simple consumption-based asset pricing model (with CRRA $\gamma = 15$). The standard two-stage GMM estimation yields the following stationary one-period transition probability matrix in the physical measure and marginal utilities for the underlying model,²²

$$\mathbf{P} = \begin{bmatrix} 0.0425 & 0.4847 & 0.4727 \\ 0.0385 & 0.5840 & 0.3775 \\ 0.0192 & 0.5799 & 0.4009 \end{bmatrix}, \begin{bmatrix} M_1 \\ M_2 \\ M_3 \end{bmatrix} = \begin{bmatrix} 4.7789 \\ 0.9029 \\ 0.6247 \end{bmatrix}, \qquad \delta = 0.98, \qquad (36)$$

where recall that the value of the time discount factor δ is exogenously set in the thought experiment. Several observations on the stylized features of the calibrated underlying model are in order. First, starting from any current state $\{i\} \in \{1, 2, 3\}$, the chance of reaching state $\{1\}$ in the next period is much smaller than reaching the other two states, $p_{i1} \ll p_{i2}, p_{i2},$ $\forall \{i\} \in \{1, 2, 3\}$. This characterization of the transition probabilities quantifies $\{1\}$ as a rare state. Second, the marginal utility associated with state $\{1\}$ is significantly higher than that associated with $\{2\}$, which in turn is higher than that associated with $\{3\}$, or $M_1 \gg M_2 > M_3$. This characterization of the marginal utility ordering quantifies $\{1\}$ as a disaster, $\{2\}$ as a normal, and $\{3\}$ as a boom state. Third, the implied one-period AD asset matrix **A**, whose i, j-entry is $A_{ij} = \delta \frac{M_i}{M_i} p_{ij}, \{i\}, \{j\} \in \{1, 2, 3\}$, reflects the combined effect of transition probabilities and marginal utilities,

$$\mathbf{A} = \begin{bmatrix} 0.0417 & 0.0898 & 0.0606 \\ 0.1999 & 0.5723 & 0.2559 \\ 0.1441 & 0.8215 & 0.3929 \end{bmatrix}.$$
 (37)

Notably, starting from any current state $\{i\} \in \{1, 2, 3\}$, the AD contract that pays off if the disaster state realizes next period is significantly cheaper than contracts that pays off if other two states realize, $A_{i1} \ll A_{i2}, A_{i3}, \forall \{i\} \in \{1, 2, 3\}$. Intuitively, while state $\{1\}$ is undesirable (i.e., the disaster state with a low consumption and a high marginality), its likelihood to

²²Recall that the marginal utilities are determined only up to a multiplicative constant.

realize next period is very small (regardless of any current state). As a result, the required premium A_{i1} , $\forall \{i\} \in \{1, 2, 3\}$, to insure against such an adverse but highly unlikely event is low. In other words, the exceedingly rare incidence of a disaster state sufficiently negates the severity of that state in the underlying (calibrated) model, resulting in sufficiently low underlying disaster state prices $\{A_{i1}\}$. These calibrated features are not only characteristic to the U.S. consumption dynamics but also crucial to explain the recovery inconsistency direction in our setting as we discuss below.

Consolidation: Once the calibration to the U.S. economy has been constructed, we take the calibration results (36) as the underlying (but unobserved) market model S. We examine the recovery process associated with an analyst's subjective specification \overline{S} , which is related to the underlying S by the following consolidation scheme

$$\begin{cases} \text{Underlying specification:} \\ \mathcal{S} = \{1, 2, 3\} \end{cases} \longrightarrow \begin{cases} \text{Subjective specification:} \\ \overline{\mathcal{S}} = \{\overline{1}, \overline{2}\} \text{ with: } \{\overline{1}\} = \{1, 2\}, \ \{\overline{2}\} = \{3\}. \end{cases} \end{cases}$$

$$(38)$$

Note that the calibration results (36) allow us to find the AD prices $A_{\tau;ij}$ initiated on the state $\{i\}$ today and maturing in τ periods in state $\{j\}$, for all states $\{i\}, \{j\} \in S$ and all periods τ .²³ Below, we analyze the direction of recovery inconsistency in the subjective specification \overline{S} when the current state is either a coupled state or a single state.

4.2 **Recovery Results and Discussion**

To identify the economic features of the subjective specification that are relevant for the associated recovery inconsistency direction, we analyze and contrast two alternative and

²³In fact, assuming the underlying specification S is known and suppressing all information about the underlying time discount factor and marginal utilities, the employment of these AD prices $A_{\tau;ij}$ alone suffices to pin down the full one-period AD matrix $\mathbf{A} = (\mathbf{A}_{\tau})^{-1} \mathbf{A}_{\tau+1}$ (4), (15), which then uniquely recovers the underlying model by solving the dominant eigenvalue and eigenvector, $\mathbf{A}\mathbf{x} = \delta\mathbf{x}$ (2). That is, in case the underlying (true) specification S is known, Ross (2015)'s recovery process uniquely and correctly recovers the risk and time preferences of the underlying market model. Our thought experiment is designed to address the alternative and practical challenge that the underlying specification S is unobserved and unknown to the analyst.

exhaustive scenarios, namely, the current state being a coupled state and a single state.

Current coupled state

Assume that the current underlying state is the normal state $\{2\}$ in the calibration model of the U.S. economy. As the underlying specification S is not observed, the analyst implementing the recovery process necessarily perceives the current state to be $\{\overline{1}\}$, which is the coupled state that the true underlying (but unobserved) current state $\{2\}$ belongs to (see (38)). Accordingly, this scenario is referred to as the current coupled state and illustrated in Figure 3. Given the current underlying state $\{2\}$, the consolidation scheme (38), and the

Current coupled state = $\{\overline{1}\}$ (underlying state = $\{2\}$)



Figure 3: Underlying (left panel) and consolidated (right panel) state space specifications associated with the consolidation scheme (38). The current underlying (true) state is $\{2\}$, which is perceived as the coupled state $\{\overline{1}\}$ in the subjective specification by the analyst.

current state $\{\overline{1}\}$ perceived by the analyst, the thought experiment starts with consolidating and generating the perceived AD prices $\overline{A}_{\tau;\overline{1}\overline{j}}$ from the underlying AD prices $\{A_{\tau;ij}\}$, which are observed in the setup as explained below (38). That is, $\overline{A}_{\tau;\overline{1}\overline{j}} = \sum_{\{j\}\in\{\overline{j}\}} A_{\tau;2j}$ (16), where $\{\overline{j}\} \in \overline{S} = \{\overline{1},\overline{2}\}$, and τ is the maturity of the AD asset. Then follows the oneperiod consolidated AD matrix \overline{A} (17), its dominant eigenvalue (the time discount factor) $\overline{\delta}$ and eigenvector (the inverse marginal utilities) $\overline{\mathbf{x}}$ (2), and the transition probabilities (3) recovered by the analyst,

$$\overline{\mathbf{A}} = \left(\overline{\mathbf{A}}_{\tau}\right)^{-1} \overline{\mathbf{A}}_{\tau+1}, \qquad \overline{\mathbf{A}}\overline{\mathbf{x}} = \overline{\delta}\overline{\mathbf{x}}, \qquad \overline{p}_{t,t+1}(\overline{i},\overline{j}) = \overline{\delta}^{-1} \overline{A}_{\overline{i}\overline{j}} \frac{\overline{x}_{\overline{j}}}{\overline{x}_{\overline{i}}}, \quad \forall \overline{i}, \overline{j} \in \overline{\mathcal{S}} = \{\overline{1},\overline{2}\}.$$
(39)

Recovery results: Using the numerical values (36)-(37), the analyst obtains the one-period consolidated AD matrix in the current setting

$$\overline{\mathbf{A}} = \begin{bmatrix} 0.3602 & 0.1303 \\ 1.7082 & 0.6193 \end{bmatrix}$$

and recovers the following time discount factor $\overline{\delta}$, marginal utilities $\{\overline{M}_{\overline{i}}\}$, and the one-period transition probability matrix $\overline{\mathbf{P}}$ in the subjective specification $\overline{\mathcal{S}}$

$$\overline{\delta} = 0.979, \qquad \begin{bmatrix} \overline{M}_{\overline{1}} \\ \overline{M}_{\overline{2}} \end{bmatrix} = \begin{bmatrix} 4.8525 \\ 1.0219 \end{bmatrix}, \qquad \overline{\mathbf{P}} = \begin{bmatrix} 0.3680 & 0.6320 \\ 0.3675 & 0.6325 \end{bmatrix}. \tag{40}$$

Possible inconsistencies between the analyst's recovery results (40) and the underlying (36) and their magnitude and direction can be illustrated by contrasting these numerical values with the implications of the current consolidation scheme (Figure 3). In particular,

$$\overline{\delta} - \delta = -0.001, \quad \text{and} \quad \begin{cases} \underline{\overline{p}_{t,t+1}(\overline{1},\overline{1})}_{0.3680} < \underline{p_{t,t+1}(2,1)}_{0.0385} + \underline{p_{t,t+1}(2,2)}_{0.5840}, \\ \underline{\overline{p}_{t,t+1}(\overline{1},\overline{1})}_{0.3680} < \underline{p_{t,t+1}(1,1)}_{0.0425} + \underline{p_{t,t+1}(1,2)}_{0.4847}. \end{cases}$$
(41)

Other consistency conditions are also violated, implying a systematic inconsistency between all recovered quantities in the subjective and underlying state specifications and signifying Proposition 1's generic assertion.²⁴

²⁴Adapting the consistency conditions (9), (10) to the current consolidation scheme of Figure 3, the following equations should be identically zero if no inconsistencies exist: $\overline{p}_{t,t+1}(\overline{1},\overline{1}) - (p_{t,t+1}(2,1)\frac{M_1}{M_2} + p_{t,t+1}(2,2)) = -0.42$, and $\frac{\overline{M_2}}{\overline{M_1}} - \frac{M_3}{M_2} \frac{p_{t,t+1}(2,3)}{1-p_{t,t+1}(2,1)\frac{M_1}{M_2}-p_{t,t+1}(2,2)} = -1.0209$. Evidently, the significant inconsistencies exist for both the analyst's recovered transition probabilities and the recovered

Discussion: Several observations on the comparative results (41) of the current numerical setting are in order. First, the inconsistency in the recovered time discount factor exists but is small and practically negligible. Second and in contrast, the inconsistency in the recovered transition probabilities is significant. Since the analyst perceives a current state $\{\overline{1}\}\$, the analyst's recovered probability $\overline{p}_{t,t+1}(\overline{1},\overline{1})$ to remain in this coupled state next period significantly undershoots the corresponding underlying transition probability, whether we use as benchmark the transition probability (i) $p_{t,t+1}(2,1) + p_{t,t+1}(2,2)$ associated with the current underlying state {2}, or (ii) $p_{t,t+1}(1,1) + p_{t,t+1}(1,2)$ associated with the current state {1} that is not the current underlying (true) state but might be in the analyst's perspective.²⁵ Third, the recovery process (39) is important to understand the undershooting (41) of the recovered probabilities in the subjective specification. For clarity, our intuitive discussion below concentrates on the leading component of the thought experiment process underlying (39), leaving an explicit matrix inversion analysis concerning all components to Appendix B.2. Given an earlier observation that the inconsistency in the time discount factor is practically negligible, our discussion also omits such an inconsistency, i.e., taking $\overline{\delta} \equiv \delta$ for simplicity. We now discuss the two undershooting relationships (41) of the transition probability in turn.

To demonstrate the first undershooting relationship, we relate and compare $\overline{p}_{t,t+1}(\overline{1},\overline{1})$ in the subjective specification with the underlying $p_{t,t+1}(2,2)$. The probability to remain in the current state next period is proportional to the respective one-period AD price, $p_{t,t+1}(2,2) =$ $\delta^{-1}A_{22}$ (3) and $\overline{p}_{t,t+1}(\overline{1},\overline{1}) = \delta^{-1}\overline{A}_{\overline{1}\overline{1}}$ (39). Therefore, an undershooting of the recovered probability to remain in the current state next period amounts to an undervaluation of the

marginal utilities.

²⁵Since the current underlying (true) state is $\{2\}$ while the analyst perceives a current coupled state $\{\overline{1}\} = \{1, 2\}$ (Figure 3), the analyst's perception is influenced by both the current ($\{2\}$) and non-current ($\{1\}$) underlying states. For robustness, our comparative analysis (41) considers both states $\{1\}$ and $\{2\}$ as the benchmark underlying state.

respective one-period AD price in the subjective specification

$$\overline{p}_{t,t+1}(\overline{1},\overline{1}) < p_{t,t+1}(2,2) \quad \iff \quad \overline{A}_{\overline{1}\,\overline{1}} < A_{22}. \tag{42}$$

Note that the undervaluation of one-period AD asset prices is intrinsic to the analyst's subjective specification \overline{S} because the one-period AD price matrix \overline{A} is implied (but not observed) from the τ -period (observable) AD prices via a recursive recovery (matrix) equation, $\overline{A} = (\overline{A}_{\tau})^{-1} \overline{A}_{\tau+1}$ (39). By concentrating on the leading (diagonal) components of the recursive recovery matrix equation and consolidating the observed τ -period AD prices, we can transform the undervaluation (42) of the implied one-period AD price into an equivalent (approximate but intuitive) inequality of the observed τ -period AD prices,²⁶

$$\frac{A_{\tau+1;21} + A_{\tau+1;22}}{A_{\tau;21} + A_{\tau;22}} < \frac{A_{\tau+1;22}}{A_{\tau;22}} \quad \text{or equivalently,} \quad \frac{A_{\tau+1;21}}{A_{\tau+1;22}} < \frac{A_{\tau;21}}{A_{\tau;22}} \tag{43}$$

Recall that the stationary transition probability calibrated to the underlying U.S. economy exhibits an important feature, namely, reaching the disaster state {1} next period is an highly unlikely event independent of current state *i*, or $p_{t,t+1}(i,1) \ll p_{t,t+1}(i,2), p_{t,t+1}(i,3)$, $\forall i$, (as observed below Equation (36)). As a result, the likelihood of reaching the disaster state {1} relative to reaching other states in τ periods from now tends to decline with the horizon τ , i.e., $\frac{p_{t,t+\tau+1}(2,1)}{p_{t,t+\tau+1}(2,2)} < \frac{p_{t,t+\tau}(2,1)}{p_{t,t+\tau}(2,2)}$. Consequently, the τ -period AD asset that insures against (i.e., pays off in) the disaster state is relatively less valuable as τ increases, i.e., $\frac{A_{\tau+1;21}}{A_{\tau+1;22}} < \frac{A_{\tau;21}}{A_{\tau;22}}$. This explains inequality (43), or equivalently (42), which then implies the first undershooting relationship of the recovered transition probability in (41).

To demonstrate the second undershooting relationship, we relate and compare $\overline{p}_{t,t+1}(\overline{1},\overline{1})$ in the subjective specification with the underlying $p_{t,t+1}(1,2)$. Using $p_{t,t+1}(1,2) = \delta \frac{M_1}{M_2} A_{12}$

²⁶In the leading order, the diagonal components of the matrix equation $\overline{\mathbf{A}} = (\overline{\mathbf{A}}_{\tau})^{-1} \overline{\mathbf{A}}_{\tau+1}$ implies that $\overline{A}_{\overline{1}\overline{1}} \approx \frac{\overline{A}_{\tau+1;\overline{1}\overline{1}}}{\overline{A}_{\tau;\overline{1}\overline{1}}}$. Similarly for the subjective specification, we have $A_{22} \approx \frac{A_{\tau+1;22}}{A_{\tau;22}}$. Next, the consolidation in Figure 3 of the τ -period AD asset prices, $\overline{A}_{\tau;\overline{1}\overline{1}} = A_{\tau;21} + A_{\tau;22}, \forall \tau$, implies the first inequality in (43). The second inequality in (43) arises from the first after a simple algebraic simplification.

(39) and $\overline{p}_{t,t+1}(\overline{1},\overline{1}) = \delta \overline{A}_{\overline{1}\overline{1}}$ (3), we transform the undershooting of the recovered transition probability into a respective undervaluation of the one-period AD price (similar to (42))

$$\overline{p}_{t,t+1}(\overline{1},\overline{1}) < p_{t,t+1}(1,2) \iff \overline{A}_{\overline{1}\,\overline{1}} < \frac{M_1}{M_2} A_{12}.$$
 (44)

By consolidating the τ -period AD asset prices and concentrating on the leading components of the recursive recovery matrix equation, the undervaluation (44) of the (implied) one-period AD asset price becomes an inequality involving the (observable) τ -period AD asset prices,²⁷

$$\frac{A_{\tau+1;21} + A_{\tau+1;22}}{A_{\tau;21} + A_{\tau;22}} < \frac{M_1}{M_2} \frac{A_{\tau+1;22}}{A_{\tau;21}} \quad \text{or equivalently,} \quad \frac{1 + \frac{A_{\tau+1;21}}{A_{\tau+1;22}}}{1 + \frac{A_{\tau;22}}{A_{\tau;21}}} < \frac{M_1}{M_2}. \tag{45}$$

The Euler pricing equation for the τ -period AD asset prices, $A_{\tau;ij} = \delta^{-\tau} p_{t,t+\tau}(i,j) \frac{M_j}{M_i}, \forall \tau, \{i\}$, and $\{j\}$, further transforms the above inequality into

$$1 + \frac{p_{t,t+\tau+1}(2,1)}{p_{t,t+\tau+1}(2,2)} \frac{M_1}{M_2} < \frac{M_1}{M_2} + \frac{p_{t,t+\tau}(2,2)}{p_{t,t+\tau}(2,1)}$$
(46)

In the current setting calibrated to the underlying U.S. economy, state {1} is a disaster state with elevated marginal utility, $M_1 > M_2$, implying that the first term on the LHS is smaller than the first term on the RHS of (46). The dominance of exceedingly rare disaster event over its severity in the disaster state price in the underlying model (as observed below (37)) and the declining likelihood of reaching the disaster state with horizon τ (as observed below (43)) imply that the second term on the LHS is smaller than the second term on the RHS of (46). Together, these features explain the inequality (46), or equivalently (44) and (45), which then imply the second undershooting relationship of the recovered transition probability in (41).

²⁷Similar to (43), in the leading (diagonal) order of the recursive recovery matrix equation and using the consolidation of the τ -period AD asset prices, the left-hand side (LHS) of the first inequality in (45) is $\overline{A}_{\overline{11}}$, the right-hand side (RHS) is $\frac{M_1}{M_2}A_{12}$. The second inequality in (45) arises from the first after a simple algebraic simplification.



Figure 4: This figure plots the difference between the recovered one-period transition probability $\overline{p}_{t,t+1}(\overline{1},\overline{1})$ in the subjective specification and the corresponding combined transition probabilities $[p_{t,t+1}(2,1) + p_{t,t+1}(2,2)]$ in the underlying (true) model versus the severity $\frac{M_1}{M_2}$ of the underlying disaster state {1}. The current underlying (true) state is {2}, which is perceived as the coupled state { $\overline{1}$ } in the subjective specification by the analyst as depicted in Figure 3. A negative value of the difference, $\overline{p}_{t,t+1}(\overline{1},\overline{1}) - [p_{t,t+1}(2,1) + p_{t,t+1}(2,2)] < 0$, indicates an undershooting of the transition probability recovered by the analyst.

To illustrate the robustness of the undershooting of recovered the transition probability by the responsible features of the underlying market model, we vary numerically the severity of the disaster state as quantified by the ratio of marginal utilities in disaster and normal states $\frac{M_1}{M_2}$. Figure 4 plots the difference $\bar{p}_{t,t+1}(\bar{1},\bar{1}) - [p_{t,t+1}(2,1) + p_{t,t+1}(2,2)]$ and Figure 5 the difference $\bar{p}_{t,t+1}(\bar{1},\bar{1}) - [p_{t,t+1}(1,1) + p_{t,t+1}(1,2)]$ between the one-period recovered and underlying transition probabilities against the ratio $\frac{M_1}{M_2}$. The plots show a robust undershooting of the recovered probability $\bar{p}_{t,t+1}(\bar{1},\bar{1})$ in the subjective specification for various values of the disaster state's severity, substantiating both undershooting relationships in (41) for a robust range of parameters. Figure 6 plots the difference $\bar{rx}_t^s(\bar{1}) - rx_t^s(2)$ between the recovered conditional stock market risk premium and the underlying counterpart. This difference in the conditional risk premia is computed from the underlying and recovery results and hence



Figure 5: This figure plots the difference between the recovered one-period transition probability $\overline{p}_{t,t+1}(\overline{1},\overline{1})$ in the subjective specification and the corresponding combined transition probabilities $[p_{t,t+1}(1,1) + p_{t,t+1}(1,2)]$ in the underlying (true) model versus the severity $\frac{M_1}{M_2}$ of the underlying disaster state {1}. The current underlying (true) state is {2}, which is perceived as the coupled state { $\overline{1}$ } in the subjective specification by the analyst as depicted in Figure 3. A negative value of the difference, $\overline{p}_{t,t+1}(\overline{1},\overline{1}) - [p_{t,t+1}(1,1) + p_{t,t+1}(1,2)] < 0$, indicates an undershooting of the transition probability recovered by the analyst.

also captures the recovery inconsistencies and their direction.²⁸ The robust overshooting of the stock market risk premium exhibited in Figure 6 reflects the fact that the current state $\{\overline{1}\}$ perceived by the analyst is an adverse state (while the true current state $\{2\}$ is normal state in the underlying model). This results in a high recovered current marginal utility $\overline{M}_{\overline{1}}$ (in relation to $\overline{M}_{\overline{2}}$ (40)) compared to the true current moderate marginal utility M_2 (in relation to M_1 and M_3). This suppresses the current stock price obtained by the analyst, compared to the (true) current stock price in the underlying model. The subjective undervaluation of the stock market then translates into an overshooting of the conditional risk premium of the stock market.

²⁸The recovered conditional stock market risk premium $\overline{rx}_t^s(\overline{1})$ is computed using the recovered marginal utilities and probabilities (40) in the Euler pricing equation for the two-state (consolidated) CCAPM model $\{\overline{1},\overline{2}\}$ conditional on the consolidated current coupled state $\{\overline{1}\}$. The underlying conditional stock market risk premium $rx_t^s(2)$ is computed using the marginal utilities and probabilities (36) for the underlying threestate (calibrated) CCAPM model $\{1, 2, 3\}$ conditional on the underlying (true) current state $\{2\}$.



Figure 6: This figure plots the difference between the conditional stock market risk premium $\overline{rx}_t^s(\overline{1})$ recovered using the subjective specification and its counterpart $rx_t^s(2)$ in the underlying (true) model versus the severity $\frac{M_1}{M_2}$ of the underlying disaster state {1}. The current underlying (true) state is {2}, which is perceived as the coupled state { $\overline{1}$ } in the subjective specification by the analyst as depicted in Figure 3. A positive value of the difference, $\overline{rx}_t^s(\overline{1}) - rx_t^s(2) > 0$, indicates an overshooting of the conditional stock market risk premium recovered by the analyst.

In summary, when the current state $\{2\}$ belongs to a coupled state $\{\overline{1}\}$ in the subjective specification, the analyst's perceived current state is $\{\overline{1}\}$, which is confounded with another underlying (but not current) disaster state $\{1\}$. This confounding acts to elevate the (perceived) current marginal utility, and hence, lower the (perceived) current price of the insurances (AD assets) against the (perceived) adverse state $\{\overline{1}\}$ happening at future periods τ and $\tau + 1$. But as the underlying disaster is exceedingly rare in the calibration of the U.S. economy, the likelihood of a disaster declines with horizon τ . As a result, the analyst currently perceives a cheaper insurance against the disaster happening at $\tau + 1$ than at τ , and therefore, necessarily infers an undervalued transition probability between the adverse state $\{\overline{1}\}$ at τ and at $\tau + 1$. These economic features of the setting together explain the undershooting of the one-period probability $\overline{p}_{t,t+1}(\overline{1},\overline{1})$ (41) recovered by the analyst.

Current single state

We now assume that the current underlying state is the boom state $\{3\}$ in the calibration model of the U.S. economy. Hence, the analyst implementing the recovery process necessarily perceives the current state to be the single state $\{\overline{2}\}$ because it is the state in the subjective specification \overline{S} that coincides with the true underlying state $\{3\}$ (38). Accordingly, this scenario is referred to as the current single state and illustrated in Figure 7. The recovery

 $Current single state = \{\overline{2}\} (underlying state = \{3\})$



Figure 7: Underlying (left panel) and consolidated (right panel) state space specifications associated with the consolidation scheme (38). The current underlying (true) state is $\{3\}$, which is perceived as the single state $\{\overline{2}\}$ in the subjective specification by the analyst.

process by the analyst in the thought experiment then follows the formalism specified in Equation (39).

Recovery results: Using the numerical values (36)-(37), the analyst obtains the one-period consolidated AD matrix in the current setting

$$\overline{\mathbf{A}} = \begin{bmatrix} 0.2720 & 0.1034\\ 1.9599 & 0.6960 \end{bmatrix}.$$
(47)

and recovers the following time discount factor, marginal utilities, and the one-period tran-

sition probability matrix in the subjective specification

$$\overline{\delta} = 0.9816, \qquad \begin{bmatrix} \overline{M}_{\overline{1}} \\ \overline{M}_{\overline{2}} \end{bmatrix} = \begin{bmatrix} 6.9342 \\ 1.0106 \end{bmatrix}, \qquad \overline{\mathbf{P}} = \begin{bmatrix} 0.2771 & 0.7229 \\ 0.2910 & 0.7090 \end{bmatrix}. \tag{48}$$

The magnitude and direction of inconsistencies between the analyst's recovery results (48) and the underlying (36) can be illustrated by contrasting these numerical values with the implications of the current consolidation scheme (Figure 7). In particular,

$$\overline{\delta} - \delta = 0.0016, \quad \text{and} \quad \underbrace{\overline{p}_{t,t+1}(\overline{2},\overline{2})}_{0.7090} > \underbrace{p_{t,t+1}(3,3)}_{0.4009}$$
(49)

We are interested in economic features responsible for the recovery inconsistencies and their directions.

Discussion: The results reported in (49) for the current single state show the inconsistency in both the recovered time discount factor and transition probabilities. While the time discount factor's inconsistency is small and practically negligible, the inconsistency in the recovered transition probabilities is significant, similar to the findings (41) for the current coupled state. However, in contrast to (41), the analyst's recovered probability $\bar{p}_{t,t+1}(\bar{2},\bar{2})$ to remain in the current state $\{\bar{2}\}$ next period significantly overshoots the (true) underlying transition probability $p_{t,t+1}(3,3)$ in (49). This change in the recovery inconsistency direction indicates the importance of the current state (being single or coupled) in the recovery process. In the intuitive analysis below, we again omit the practically negligible inconsistency in the time discount factor and concentrate on the leading component of the analyst's recovery process (39) for simplicity (leaving the technical analysis concerning all components to Appendix B.2).

To demonstrate the overshooting of the recovered probability in the subjective specification (49), we relate it to the overvaluation of the respective one-period AD price. Since $p_{t,t+1}(3,3) = \delta^{-1}A_{33}$ (3) and $\overline{p}_{t,t+1}(\overline{2},\overline{2}) = \delta^{-1}\overline{A}_{\overline{2}\overline{2}}$ (39), we have

$$\overline{p}_{t,t+1}(\overline{2},\overline{2}) > p_{t,t+1}(3,3) \quad \iff \quad \overline{A}_{\overline{2}\overline{2}} > A_{33}.$$

$$(50)$$

Recall that the one-period AD prices are implied from the recursive recovery equation systems involving the τ -period AD prices

$$\overline{A}_{\tau+1;\overline{2}\,\overline{2}} = \overline{A}_{\tau;\overline{2}\,\overline{1}}\overline{A}_{\overline{1}\,\overline{2}} + \overline{A}_{\tau;\overline{2}\,\overline{2}}\overline{A}_{\overline{2}\,\overline{2}}, \qquad A_{\tau+1;33} = A_{\tau;31}A_{13} + A_{\tau;32}A_{23} + A_{\tau;33}A_{33}.$$
(51)

Given that the current state is a single state, its underlying and subjective specifications coincide, $\{\overline{2}\} \equiv \{3\}$. Hence the analyst correctly observe τ -period AD prices for all lengths τ , i.e., $\overline{A}_{\tau;\overline{2}\overline{2}} = A_{\tau;33}$ and $\overline{A}_{\tau+1;\overline{2}\overline{2}} = A_{\tau+1;33}$. The substitution of these identities into the two recursive pricing equations in (51) produces an equivalent condition for the overvaluation of the implied one-period AD price (50), $\overline{A}_{\overline{2}\overline{2}} > A_{33} \iff \overline{A}_{\tau;\overline{2}\overline{1}}\overline{A}_{\overline{1}\overline{2}} < A_{\tau;31}A_{13} + A_{\tau;32}A_{23}$. Using the consolidation scheme (38), $\overline{A}_{\tau;\overline{2}\overline{1}} = A_{\tau;31} + A_{\tau;32}$, we can express this condition for the overvaluation of the implied AD price into an equivalent inequality involving the observed τ -period AD prices,

$$\overline{A}_{\overline{1}\,\overline{2}} < \frac{A_{\tau;31}A_{13} + A_{\tau;32}A_{23}}{A_{\tau;31} + A_{\tau;32}} = \frac{A_{\tau;31}A_{13} + A_{\tau;32}A_{23}}{A_{\tau;32}\left(1 + \frac{A_{\tau;31}}{A_{\tau;32}}\right)}.$$
(52)

That is, a complementarity between the implied one-period AD prices $\overline{A}_{\overline{2}\overline{2}}$ (overvalued) and $\overline{A}_{\overline{1}\overline{2}}$ (undervalued) exists to make sure that the analyst correctly observes the τ -period AD prices, $\overline{A}_{\tau;\overline{2}\overline{2}} = A_{\tau;33}$, for all τ (51). The dominance of exceedingly rare disaster event over its severity in the disaster state price in the underlying model (as observed below (37)) and the declining likelihood of reaching the disaster state {1} with horizon τ (as observed below (43)) results in a low price for AD assets paying off in the disaster state, $\frac{A_{\tau;31}}{A_{\tau;32}} \ll 1$. Therefore, in

the leading order, inequality simplifies to

$$\overline{A}_{\overline{1}\,\overline{2}} < \frac{A_{\tau;31}A_{13} + A_{\tau;32}A_{23}}{A_{\tau;32}} = A_{23} + \frac{A_{\tau;31}}{A_{\tau;32}}A_{13} \approx A_{23}.$$
(53)

Note that because the coupled state $\{\overline{1}\} = \{1, 2\}$ perceived by the analyst contains the underlying (true) disaster state $\{1\}$, the price $\overline{A}_{\overline{12}}$ of the AD asset initiated on that state $\{\overline{1}\}$ is relatively low due to the high current marginal utility,²⁹ explaining inequality (53). In the numerical calibration (see (37) and (47)), the inequality indeed holds, $0.1034 = \overline{A}_{\overline{12}} < A_{23} = 0.2559$. As a result, it also explains the equivalent undershooting of $\overline{A}_{\overline{12}}$ (52), the overshooting of the recovered probability in the subjective specification (50) and then (49).



Figure 8: This figure plots the difference between the recovered one-period transition probability $\overline{p}_{t,t+1}(\overline{2},\overline{2})$ in the subjective specification and the corresponding transition probability $p_{t,t+1}(3,3)$ in the underlying (true) model versus the severity $\frac{M_1}{M_2}$ of the underlying disaster state $\{1\}$. The current underlying (true) state is $\{3\}$, which is perceived as the single state $\{\overline{2}\}$ in the subjective specification by the analyst as depicted in Figure 7. A positive value of the difference, $\overline{p}_{t,t+1}(\overline{2},\overline{2}) - p_{t,t+1}(3,3) > 0$, indicates an overshooting of the transition probability recovered by the analyst.

To illustrate the robustness of the overshooting of recovered the transition probability by

²⁹As $\overline{A}_{\overline{1}\overline{2}} \sim \frac{M_{\overline{2}}}{\overline{M}_{\overline{1}}}$, the AD asset $\overline{A}_{\overline{1}\overline{2}}$ is inexpensive because it insures against (i.e., pays off in) the boom state $\overline{2}$ tomorrow, the state in which the analyst is well-off (i.e., having a relatively low marginal utility $\overline{M}_{\overline{2}}$).



Figure 9: This figure plots the difference between the conditional stock market risk premium $\overline{rx}_t^s(\overline{2})$ recovered using the subjective specification and its counterpart $rx_t^s(3)$ in the underlying (true) model versus the severity $\frac{M_1}{M_2}$ of the underlying disaster state {1}. The current underlying (true) state is {3}, which is perceived as the single state { $\overline{2}$ } in the subjective specification by the analyst as depicted in Figure 7. A positive value of the difference, $\overline{rx}_t^s(\overline{2}) - rx_t^s(3) > 0$, indicates an overshooting of the conditional stock market risk premium recovered by the analyst.

the responsible features of the underlying market model, we vary numerically the severity of the disaster state as quantified by the ratio of marginal utilities in disaster and normal states $\frac{M_1}{M_2}$. Figure 8 plots the difference $\overline{p}_{t,t+1}(\overline{2},\overline{2}) - p_{t,t+1}(3,3)$ between the one-period recovered and underlying transition probabilities against the ratio $\frac{M_1}{M_2}$. The plot shows a robust overshooting of the recovered probability $\overline{p}_{t,t+1}(\overline{2},\overline{2})$ in the subjective specification for various values of the disaster state's severity, substantiating the overshooting relationship (49) for a robust range of parameters. Figure 9 plots the difference $\overline{rx}_t^s(\overline{2}) - rx_t^s(3)$ between the recovered conditional stock market risk premium and the underlying counterpart.³⁰ Similar to Figure 6, the difference in conditional stock market risk premia across the recovered and un-

³⁰Per Figure 7, the recovered conditional stock market risk premium $\overline{rx}_t^s(\overline{2})$ is computed using the recovered marginal utilities and probabilities (48) for the two-state (consolidated) CCAPM model { $\overline{1}, \overline{2}$ } conditional on the consolidated current coupled state { $\overline{2}$ }. The underlying conditional stock market risk premium $rx_t^s(3)$ is computed using the marginal utilities and probabilities (36) in the Euler equation for the underlying three-state (calibrated) CCAPM model {1, 2, 3} conditional on the underlying (true) current state {3}.

derlying models are results of the recovery inconsistencies (the relative magnitude of current marginal utilities and transition probabilities starting from current states in consolidated and underlying models, in particular).

In summary, when the current state is a single boom state {3}, the analyst's subjective specification of the current state { $\overline{2}$ } is correct, i.e., { $\overline{2}$ } = {3}. This assures that the analyst correctly observes the current price of the insurance against the event of remaining in the boom state at $\tau + 1$, or $\overline{A}_{\tau+1;\overline{2}\overline{2}} = A_{\tau+1;33}$. The U.S. economy calibrated model's exceedingly rare disaster state features a declining disaster's insurance price with the horizon, $\overline{A}_{\tau+1;\overline{2}\overline{1}} < \overline{A}_{\tau;\overline{2}\overline{1}}$. Therefore, to counter this outsized contribution of the τ -period insurance price $\overline{A}_{\tau;\overline{2}\overline{1}}$ to the correctly observed $\tau + 1$ -period contingent price $\overline{A}_{\tau+1;\overline{2}\overline{2}}$, the analyst necessarily infers an undervalued one-period (from τ to $\tau+1$) asset price $\overline{A}_{\overline{1}\overline{2}}$, or equivalently, an overvalued complementary price $\overline{A}_{\overline{2}\overline{2}}$.³¹ The latter asset overvaluation amounts to an overshooting of the corresponding probability $\overline{p}_{t,t+1}(\overline{2},\overline{2})$ recovered by the analyst. These features together therefore explain the overshooting of the one-period transition probability (49).

5 Conclusion

This paper investigates the implementability aspects of a consistent recovery of time and risk preferences and the state probability distribution in the physical measure from asset prices. The recovery process requires a subjective input specification of the state space since such a specification is not observed prior to the recovery implementation. Different input specifications therefore lead to different recovery results that are mutually inconsistent unless the underlying market model satisfies a strong necessary and sufficient condition. The

³¹First, note that the contribution of the τ -period AD asset price $\overline{A}_{\tau;\overline{2}\,\overline{1}}$ to the τ +1-period AD asset price $\overline{A}_{\tau+1;\overline{2}\,\overline{2}}$ is via the product $\overline{A}_{\tau;\overline{2}\,\overline{1}}\overline{A}_{\overline{1}\,\overline{2}}$ (see the first recursive equation in (51)). When $\overline{A}_{\tau+1;\overline{2}\,\overline{2}}$ is correctly observed, a (perceived) overvaluation of $\overline{A}_{\tau;\overline{2}\,\overline{1}}$ by an analyst is necessarily accompanied (and countered) by a (perceived) undervaluation of $\overline{A}_{\overline{1}\,\overline{2}}$ by the same analyst. Second, the complementarity of the perceived AD prices $\overline{A}_{\overline{1}\,\overline{2}}$ is observed below Equation (52).

inconsistency in the recovery results arises because the transition between different input specifications induces inadvertent but irreversible losses of information in the price data. As a result, the recovery consistency issue prevails in the presence of perfect (unlimited and error-free) price data and sophisticated input specifications. In the limit of a continuous state space specification, the inconsistency of the recovery results persists and can be seen as a result of an improper subjective discretization of the state space. A model calibrated to the U.S. economy that features a stylized rare disaster state helps to elucidate the direction (overshooting vs. undershooting) of the recovery results. Extensions to the original recovery framework, such as the generalized and best-fit recovery approaches, do not solve the recovery consistency issue because their implementation also requires a subjective input specification for the underlying state space. These findings indicate an elusive nature of implementing a consistent recovery process.

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Internet Appendices

(not intended for publication)

These online appendices provides supporting details and technical derivations for the findings and analysis of the main text. Appendix A presents a proof of Proposition 1. Appendix B addresses the inconsistency and its direction in the recovery; Appendix B.1 concerns the perturbative analysis, and Appendices B.2 and B.3 the proof and calibration of the direction of the recovery inconsistency. Appendix C addresses the extensions to the basic recovery; Appendix C.1 concerns the generalized recovery and Appendix C.2 the best-fit implementation of both the basic and generalized recoveries.

A Consistency Conditions in Recovery

This appendix present a proof of Proposition 1. The proof addresses separately whether the current state is a single or a coupled state.

Case 1 - Single Current State: We consider an original specification of $S = \{1, \dots, S\}$ and a consolidated specification $\overline{S} = \{\overline{1}, \overline{2}, \dots, \overline{K}, \overline{S}\}$, which are adopted by two analysts. The mapping (or consolidation scheme) between the two specifications is as follows: $\{\overline{1}\} = \{1\}, \{\overline{2}\} =$ $\{2\}, \dots, \{\overline{K}\} = \{K\}, \text{ and } \{\overline{S}\} = \{K+1, \dots, S\}$. Suppose that the current state is the single state $\{\overline{1}\}$ for the second (consolidated) analyst, and $\{1\}$ for the first (original) analyst.

In the first direction of the proof (i.e., proving the sufficient condition in (21)), we assume consistent recoveries under both specifications. As a result, the following no-arbitrage conditions on observed price data must be satisfied for the two consistent specifications:

$$A_{\tau+1;1i} = \sum_{j=\{1\}}^{\{S\}} A_{\tau;1j} A_{ji}, \quad \forall \{i\} \in \mathcal{S} \text{ and } \forall \tau \in \{1, 2, 3, \cdots\}$$
(54)

$$\overline{A}_{\tau+1;\overline{1}\,\overline{i}} = \sum_{\{\overline{j}\}=\{\overline{1}\}}^{\{\overline{S}\}} \overline{A}_{\tau;\overline{1}\,\overline{j}} \overline{A}_{\overline{j}\,\overline{i}}, \quad \forall\{\overline{i}\}\in\overline{\mathcal{S}} \text{ and } \forall\tau\in\{1,2,3,\cdots\}$$
(55)

$$\overline{A}_{\tau;\overline{1}\,\overline{i}} = \sum_{\{j\}\subset\{\overline{i}\}} A_{\tau;1j}, \quad \forall\{\overline{i}\}\in\overline{\mathcal{S}} \text{ and } \forall\tau\in\{1,2,3,\cdots\}.$$
(56)

The above equation system holds for any horizon τ in the future. However, AD price matrices **A** and $\overline{\mathbf{A}}$ contain a fixed number of entries to be solved for, resulting in an overidentified equation system. In particular, we substitute (56) into (55) and obtain

$$\sum_{\{j\}\subset\{\overline{i}\}} A_{\tau+1;1j} = \sum_{\{\overline{j}\}=\{\overline{1}\}}^{\{\overline{S}\}} \left(\sum_{\{k\}\subset\{\overline{j}\}} A_{\tau;1k}\right) \overline{A}_{\overline{j}\,\overline{i}}, \quad \forall\{\overline{i}\}\in\overline{\mathcal{S}} \text{ and } \forall\tau\in\{1,2,3,\cdots\}.$$
(57)

The equation system, (54) and (57), has an infinite number of equations but only $S^2 + (K+1)^2$ unknowns. Hence, in order for the system to have a solution, the entries in $\overline{\mathbf{A}}$ must satisfy

$$\overline{A}_{\overline{i}\,\overline{j}} = \sum_{\{j\}\subset\{\overline{j}\}} A_{ij}, \quad \forall\{\overline{i}\} = \{i\} \in \{1,\cdots,K\} \text{ and } \{\overline{j}\} \in \overline{\mathcal{S}}$$

$$(58)$$

$$\overline{A}_{\overline{S}\overline{j}} = \sum_{\{j\} \subset \{\overline{j}\}} A_{ij}, \quad \forall \{i\} \subset \{\overline{S}\} \text{ and } \{\overline{j}\} \in \overline{\mathcal{S}},$$
(59)

so that by summing up the equations (54) in states $\{i\} \subset \overline{S}$ we obtain (57).

Given that the AD price matrix **A** of the original specification S satisfies (58) and (59), the eigenvector, $\mathbf{x} = [x_1, \dots, x_S]'$, associated with the largest and positive eigenvalue δ will satisfy the following form:

$$\begin{bmatrix} A_{11} & \dots & A_{1K} & A_{1,K+1} & \dots & A_{1S} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ A_{K1} & \dots & A_{KK} & A_{K,K+1} & \dots & A_{KS} \\ A_{K+1,1} & \dots & A_{K+1,K} & A_{K+1,K+1} & \dots & A_{K+1,S} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ A_{S1} & \dots & A_{SK} & A_{S,K+1} & \dots & A_{SS} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_K \\ x_{K+1} = \overline{x}_{\overline{S}} \\ \vdots \\ x_S = \overline{x}_{\overline{S}} \end{bmatrix} = \delta \begin{bmatrix} x_1 \\ \vdots \\ x_K \\ x_{K+1} = \overline{x}_{\overline{S}} \\ \vdots \\ x_S = \overline{x}_{\overline{S}} \end{bmatrix}, \quad (60)$$

which implies the marginal utilities satisfy $M_i = M_k$, for all $\{i\}$ and $\{k\}$ belonging to the same coupled state. By the formula of recovered probabilities (3), it is easy to show that $p_{i\overline{h}} = p_{k\overline{h}}, \forall \{i\}, \{k\} \subset \{\overline{j}\}, \{\overline{h}\} \in \overline{S}$.

Next, we prove the other direction (i.e., proving the necessary condition in (21)). We assume that the inputs of the original specification satisfy $M_i = M_k, p_{i\bar{h}} = p_{k\bar{h}}, \forall \{i\}, \{k\} \subset \{\bar{j}\}, \{\bar{j}\}, \{\bar{h}\} \in \{\bar{j}\}, \{\bar{j}\}, \{\bar{h}\} \in \{\bar{j}\}, \{\bar{j}\},$

 $\overline{\mathcal{S}}$. According to (3), the original AD price matrix **A** satisfy

$$\sum_{\{j\}\subset\{\overline{j}\}} A_{ij} = \sum_{\{j\}\subset\{\overline{j}\}} A_{kj}, \quad \forall\{i\}, \{k\}\subset\{\overline{S}\} \text{ and } \{\overline{j}\}\in\overline{\mathcal{S}}.$$
(61)

Since our primary analysis concerns consistency, we assume without loss of generality that S is the true model. Consequently, we have $\mathbf{A}_{\tau+1} = \mathbf{A}_{\tau}\mathbf{A}$ holds for all horizon τ . No-arbitrage restriction of trade assets gives (56). These conditions together with (61) imply that $\overline{\mathbf{A}}_{\tau+1} = \overline{\mathbf{A}}_{\tau}\overline{\mathbf{A}}$ also holds for all τ and thus we obtain (58) and (59). Consequently, applying the recovery equation (3), we have

$$\overline{\delta} = \delta; \qquad \overline{M}_{\overline{j}} = \overline{M}_{\overline{1}}, \ \forall \{\overline{j}\} = \{j\} \in \{1, \cdots, K\};$$

$$\overline{p}_{t,t+1}(\overline{1}, \overline{j}) = p_{t,t+1}(1, j), \ \forall \{\overline{j}\} = \{j\} \in \{1, \cdots, K\}, \ \forall t; \qquad (62)$$

$$\overline{p}_{t,t+1}(\overline{1}, \overline{S}) = \sum_{\{j\}=\{K+1\}}^{\{S\}} p_{t,t+1}(1, j), \ \forall t; \qquad \overline{M}_{\overline{S}} = \sum_{\{j\}=\{K+1\}}^{\{S\}} \frac{p_{t,t+1}(1, j)}{\sum_{\{i\}=\{K+1\}}^{\{S\}} p_{t,t+1}(1, i)} \frac{M_j}{M_1}, \ \forall t.$$

That is, all consistency conditions are satisfied, establishing the recovery consistency for the case of single current state.

Case 2 - Coupled current state: We consider an original specification of $S = \{1, \dots, S\}$ and a consolidated specification $\overline{S} = \{\overline{1}, \overline{K+1}, \dots, \overline{S-1}, \overline{S}\}$, which are adopted by two analysts. The mapping (or consolidation scheme) between the two specifications is as follows: $\{\overline{1}\} = \{1, \dots, K\}, \{\overline{K+1}\} = \{K+1\}, \{\overline{K+2}\} = \{K+2\}, \dots \{\overline{S-1}\} = \{S-1\}$, and $\{\overline{S}\} = \{S\}$. Suppose that the current state is the coupled state $\{\overline{1}\}$ for the second (consolidated) analyst and $\{1\}$ for the first (original) analyst.

In the first direction of the proof (i.e., proving the sufficient condition in (21)), we assume consistent recoveries under both specifications. As a result, the same no-arbitrage conditions (54)– (56) for Case 1 on observed price data must be satisfied for the two consistent specifications. We again obtain the equation system (54) and (57). In order for the system to have a solution, the entries in $\overline{\mathbf{A}}$ must satisfy

$$\overline{A}_{\overline{1}\,\overline{j}} = \sum_{\{j\}\subset\{\overline{j}\}} A_{ij}, \quad \forall\{i\}\subset\{\overline{1}\} \text{ and } \{\overline{j}\}\in\overline{\mathcal{S}},\tag{63}$$

$$\overline{A}_{\overline{i}\overline{j}} = \sum_{\{j\}\subset\{\overline{j}\}} A_{ij}, \quad \forall\{\overline{i}\} = \{i\} \in \{K+1,\cdots,S\} \text{ and } \{\overline{j}\} \in \overline{\mathcal{S}},$$
(64)

so that by summing up the equations (54) in states $\{i\} \subset \{\overline{1}\}$ we obtain (57).

Given that the AD price matrix **A** of the original specification S satisfies (63) and (64), the eigenvector, $\mathbf{x} = [x_1, \dots, x_S]'$, associated with the largest and positive eigenvalue δ will satisfy the following form:

$$\begin{bmatrix} A_{11} & \dots & A_{1K} & A_{1,K+1} & \dots & A_{1S} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ A_{K1} & \dots & A_{KK} & A_{K,K+1} & \dots & A_{KS} \\ A_{K+1,1} & \dots & A_{K+1,K} & A_{K+1,K+1} & \dots & A_{K+1,S} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ A_{S1} & \dots & A_{SK} & A_{S,K+1} & \dots & A_{SS} \end{bmatrix} \begin{bmatrix} x_1 = \overline{x}_{\overline{1}} \\ \vdots \\ x_K = \overline{x}_{\overline{1}} \\ x_{K+1} \\ \vdots \\ x_S \end{bmatrix} = \delta \begin{bmatrix} x_1 = \overline{x}_{\overline{1}} \\ \vdots \\ x_K = \overline{x}_{\overline{1}} \\ x_{K+1} \\ \vdots \\ x_S \end{bmatrix}$$
(65)

which implies the marginal utilities satisfy $M_i = M_k$, for all $\{i\}$ and $\{k\}$ belonging to the same coupled state. By the formula of recovered probabilities (3), it is easy to show that $p_{i\bar{h}} = p_{k\bar{h}}, \forall \{i\}, \{k\} \subset \{\bar{j}\}, \{\bar{j}\}, \{\bar{h}\} \in \overline{S}$.

Next, we prove the other direction (i.e., proving the necessary condition in (21)). We assume that the inputs of the original specification satisfy $M_i = M_k, p_{i\overline{h}} = p_{k\overline{h}}, \forall \{i\}, \{k\} \subset \{\overline{j}\}, \{\overline{j}\}, \{\overline{h}\} \in \overline{S}$. According to (3), the original AD price matrix **A** satisfy

$$\sum_{\{j\}\subset\{\overline{j}\}} A_{ij} = \sum_{\{j\}\subset\{\overline{j}\}} A_{kj}, \quad \forall\{i\}, \{k\}\subset\{\overline{1}\} \text{ and } \{\overline{j}\}\in\overline{\mathcal{S}}.$$
(66)

Since our primary analysis concerns consistency, we assume without loss of generality that S is the true model. Consequently, we have $\mathbf{A}_{\tau+1} = \mathbf{A}_{\tau}\mathbf{A}$ holds for all horizon τ . No-arbitrage restriction of traded assets gives (56). These conditions together with (66) imply that $\overline{\mathbf{A}}_{\tau+1} = \overline{\mathbf{A}}_{\tau}\overline{\mathbf{A}}$

also holds for all τ and thus we obtain (63) and (64). Consequently, applying the recovery equation (3), we have

$$\begin{split} \overline{\delta} &= \delta; \qquad \overline{p}_{t,t+1}(\overline{1},\overline{1}) = \sum_{\{j\}=\{1\}}^{\{K\}} p_{t,t+1}(1,j), \ \forall t; \\ \overline{p}_{t,t+1}(\overline{1},\overline{j}) &= p_{t,t+1}(1,j) \frac{1 - \sum_{\{i\}=\{1\}}^{\{K\}} p_{t,t+1}(1,i)}{1 - \sum_{\{i\}=\{1\}}^{\{K\}} p_{t,t+1}(1,i)}, \ \forall \{\overline{j}\} = \{j\} \in \{K+1,\cdots,S\}, \ \forall t; \qquad (67) \\ \frac{\overline{M}_{\overline{j}}}{\overline{M}_{\overline{1}}} &= \frac{M_j}{M_1} \frac{1 - \sum_{\{i\}=\{1\}}^{\{K\}} p_{t,t+1}(1,i)}{1 - \sum_{\{i\}=\{1\}}^{\{K\}} p_{t,t+1}(1,i)}, \ \forall \{\overline{j}\} = \{j\} \in \{K+1,\cdots,S\}, \ \forall t. \end{split}$$

That is, all consistency conditions are satisfied, implying the recovery consistency for the case of coupled current state. Together with the derivation above for the case of single current state, this establishes Proposition $1 \blacksquare$

B Recovery Inconsistency and Direction

This appendix provides technical background and derivations for the recovery inconsistency analysis of the main text. Appendix B.1 concerns a perturbative formalism, Appendix B.2 the derivation of the inconsistency direction of the recovery, and Appendix B.3 the calibration details.

B.1 Perturbative Results: Derivations

This appendix derives Section 3.3's results by relating the perturbative recovery equation system to Vandermonde matrix. This connection yields explicit expressions for the perturbative recovered quantities in the original and consolidated specifications (with any numbers of states).

Recall that Section 3.3's perturbative setting features an underlying (true but unobserved) model {1,2,3,4} associated with marginal utilities (26) and the current (true) state {1}. Two analysts perceive subjective (simple) model a and (sophisticated) model b specified in (25) with respective consolidation schemes { $\overline{1}^a$ } = {1}, { $\overline{2}^a$ } = {2,3,4} and { $\overline{1}^b$ } = {1}, { $\overline{2}^b$ } = {2}, { $\overline{3}^b$ } = {3,4}. Recovery results of models a and b are related via their relations to the underlying model. The derivation proceeds in the following steps to obtain: (i) the perturbative component $\overline{\mathbf{B}}_{\tau}^{i}$ and unperturbed component $\overline{\mathbf{A}}_{\tau}^{i}$ of the τ -period AD price matrix (28), (ii) the inverse matrix $(\overline{\mathbf{A}}_{\tau}^{i})^{-1}$ (29) to be used in (31), and (iii) the perturbative recovered time discount factors $\delta_{1}^{i}(\varepsilon)$, $i \in \{a, b\}$, (32), (33).

<u>Step (i)</u>: The τ -period AD asset prices in the (simple) model a arises from the underlying τ -period AD asset prices via no-arbitrage relationships,

$$\overline{A}_{\tau;\overline{1}\,\overline{1}} = A_{\tau;11} = \delta^{\tau} p_{t,t+\tau}(1,1),$$

$$\overline{A}_{\tau;\overline{1}\,\overline{2}} = A_{\tau;12} + A_{\tau;13} + A_{\tau;14} = \delta^{\tau} \left(p_{t,t+\tau}(1,2)M_2 + p_{t,t+\tau}(1,3)M_3 + p_{t,t+\tau}(1,4)M_4 \right)$$
$$= \delta^{\tau} \left(p_{t,t+\tau}(1,2) + p_{t,t+\tau}(1,3) + p_{t,t+\tau}(1,4) \right) M_2 + \varepsilon \delta^{\tau} \left(2p_{t,t+\tau}(1,3) + 3p_{t,t+\tau}(1,4) \right).$$

Stacking these relationships into the matrix form give rise to the perturbative expansion $\overline{\mathbf{A}}_{\tau}^{a}(\varepsilon) = \overline{\mathbf{A}}_{\tau}^{a} + \varepsilon \overline{\mathbf{B}}_{\tau}^{a}$ in (28), where the perturbative component $\overline{\mathbf{B}}_{\tau}^{a}$ of the τ -period AD price matrix $\overline{\mathbf{A}}_{\tau}^{a}$ for analyst a is

$$\overline{\mathbf{B}}_{\tau}^{a} = \begin{bmatrix} 0 & \delta(2p_{t,t+1}(1,3) + 3p_{t,t+1}(1,4)) \\ 0 & \delta^{2}(2p_{t,t+2}(1,3) + 3p_{t,t+2}(1,4)) \\ \vdots & \vdots \\ 0 & \delta^{\tau}(2p_{t,t+\tau}(1,3) + 3p_{t,t+\tau}(1,4)) \end{bmatrix}.$$
(68)

Similarly for model *b*, the no-arbitrage relationships and the perturbative expansion $\overline{\mathbf{A}}_{\tau}^{b}(\varepsilon) = \overline{\mathbf{A}}_{\tau}^{b} + \varepsilon \overline{\mathbf{B}}_{\tau}^{b}$ (28) have the following explicit expressions

$$\overline{A}_{\tau;\overline{1}\,\overline{1}} = A_{\tau;11} = \delta^{\tau} p_{t,t+\tau}(1,1), \qquad \overline{A}_{\tau;\overline{1}\,\overline{2}} = A_{\tau;11} = \delta^{\tau} p_{t,t+\tau}(1,2),$$

 $\overline{A}_{\tau;\overline{1}\,\overline{3}} = A_{\tau;13} + A_{\tau;14} = \delta^{\tau} \left(p_{t,t+\tau}(1,3) + p_{t,t+\tau}(1,4) \right) M_2 + \varepsilon \delta^{\tau} \left(2p_{t,t+\tau}(1,3) + 3p_{t,t+\tau}(1,4) \right) .$

and

$$\overline{\mathbf{B}}_{\tau}^{b} = \begin{bmatrix} 0 & 0 & \delta(2p_{t,t+1}(1,3) + 3p_{t,t+1}(1,4)) \\ 0 & 0 & \delta^{2}(2p_{t,t+2}(1,3) + 3p_{t,t+2}(1,4)) \\ \vdots & \vdots & \vdots \\ 0 & 0 & \delta^{\tau}(2p_{t,t+\tau}(1,3) + 3p_{t,t+\tau}(1,4)) \end{bmatrix}.$$
(69)

Let $\overline{A}_{\tau}^{i}(t,:)$ denote the *t*-th row of the unperturbed τ -period (unperturbed) AD price matrix $\overline{\mathbf{A}}_{\tau}^{i}$, $i \in \{a, b\}$ in (31). The definition (4) of the τ -period AD price matrix shows that its *t*-th row contains the prices of \overline{S}^{i} AD assets initiated on the current state $\{\overline{1}^{i}\} = \{1\}$ and paying off in one of states $\{1, \ldots, \overline{S}^{i}\}$ in *t* periods $(t \leq \tau)$. Since the left eigenvectors $\{\overline{\mathbf{w}}_{k}^{i}\}, k \in \{1, \ldots, \overline{S}^{i}\}$ (31), span the \overline{S}^{i} -dimensional vector space, we look for the 1-st row vector $\overline{A}_{\tau}^{i}(1,:)$ in the following form,¹

$$\overline{A}_{\tau}^{i}(1,:) = \sum_{\{k\}=\{1\}}^{\{\overline{S}^{i}\}} \alpha_{k}^{i} \overline{\mathbf{w}}_{k}^{i'}, \qquad i \in \{a,b\}.$$
(70)

Recall that entries in the row $\overline{A}^{i}_{\tau}(1,:)$ are prices of AD assets initiated on the current state $\{\overline{1}^{i}\} = \{1\}$ and maturing next period, so $\overline{A}^{i}_{\tau}(1,:)$ is also the first row of the one-period full AD price matrix $\overline{\mathbf{A}}^{i}$. Let $\overline{\mathbf{X}}^{i}$ and $\overline{\mathbf{W}}^{i}$ denote the matrices of right and left eigenvectors the one-period AD price matrix $\overline{\mathbf{A}}^{i}$. That is, columns of $\overline{\mathbf{X}}^{i}$ are right eigenvectors, columns of $\overline{\mathbf{W}}^{i}$ (or equivalently, rows of $\overline{\mathbf{W}}^{i'}$) are left eigenvectors, of $\overline{\mathbf{A}}^{i}$. Then follows the diagonalization,

$$\overline{\mathbf{A}}^{i} = \overline{\mathbf{W}}^{i} \mathbf{Diag} \left(\delta_{1}^{i}, \dots, \delta_{\overline{S}^{i}}^{i} \right) \left(\overline{\mathbf{X}}^{i} \right)^{\prime}, \quad \text{with the normalization} \quad \overline{\mathbf{W}}^{i^{\prime}} \overline{\mathbf{X}}^{i} = \mathbb{1}_{\overline{S}^{i} \times \overline{S}^{i}}.$$
(71)

Equipped with the mutual orthogonality of left and right eigenvectors of a matrix, to determine coefficients $\{\alpha_k^i\}$, we multiply to the right of both sides of (70) by the k-th (column) right eigenvector $\overline{\mathbf{x}}_k^i$ and obtain

$$\alpha_k^i = \delta_k^i \overline{x}_{k1}^i, \qquad k \in \{1, \dots, \overline{S}^i\}, \quad i \in \{a, b\},$$
(72)

where \overline{x}_{k1}^i is the 1st-th element of the k-th right eigenvector $\overline{\mathbf{x}}_k^i$.

¹Note that the left eigenvectors $\{\overline{\mathbf{w}}_k^i\}, k \in \{1, \dots, \overline{S}^i\}$, are $\overline{S}^i \times 1$ column vectors in our notation. Hence their transposed $\overline{\mathbf{w}}_k^{i'}, k \in \{1, \dots, \overline{S}^i\}$, are $1 \times \overline{S}^i$ row vectors that span the space of \overline{S}^i -dimensional row vectors.

Next, note that the 2-nd row $\overline{A}^{i}_{\tau}(2,:)$ of the τ -period (unperturbed) AD price matrix is the 1-st row $\overline{A}^{i}_{\tau+1}(1,:)$ of the τ + 1-period AD price matrix. Hence, using the recursive equation system $\overline{\mathbf{A}}^{i}_{\tau+1} = \overline{\mathbf{A}}^{i}_{\tau}\overline{\mathbf{A}}^{i}$ (the second system in (15), which originates from (4)), the 2-nd row of the τ -period (unperturbed) AD price matrix is

$$\overline{A}_{\tau}^{i}(2,:) = \overline{A}_{\tau}^{i}(1,:)\overline{\mathbf{A}}^{i} = \left(\sum_{\{k\}=\{1\}}^{\{\overline{S}^{i}\}} \alpha_{k}^{i}\overline{\mathbf{w}}_{k}^{i'}\right)\overline{\mathbf{A}}^{i} = \sum_{\{k\}=\{1\}}^{\{\overline{S}^{i}\}} \alpha_{k}^{i}\delta_{k}^{i}\overline{\mathbf{w}}_{k}^{i'}, \qquad i \in \{a,b\},$$

because $\overline{\mathbf{w}}_{k}^{i'}$ is the k-th left eigenvector of the one-period AD price matrix $\overline{\mathbf{A}}^{i}$. Repeating this procedure yields all rows of the τ -period (unperturbed) AD price matrix,

$$\overline{A}_{\tau}^{i}(t,:) = \sum_{\{k\}=\{1\}}^{\{\overline{S}^{i}\}} \alpha_{k}^{i} \left(\delta_{k}^{i}\right)^{t-1} \overline{\mathbf{w}}_{k}^{i'}, \qquad t \in \{1,\ldots,\overline{S}^{i}\}, i \in \{a,b\},$$
(73)

with $\alpha_k^i = \delta_k^i \overline{x}_{k1}^i$, $k \in \{1, \dots, \overline{S}^i\}$, (72). Stacking rows (73) produce an explicit expression for the τ -period (unperturbed) AD price matrix $\overline{\mathbf{A}}_{\tau}^i$,

$$\overline{\mathbf{A}}_{\tau}^{i} = \underbrace{\begin{bmatrix} \alpha_{1}^{i} & \alpha_{2}^{i} & \cdots & \alpha_{\overline{S}^{i}}^{i} \\ \alpha_{1}^{i}\delta_{1}^{i} & \alpha_{2}^{i}\delta_{2}^{i} & \cdots & \alpha_{\overline{S}^{i}}^{i}\delta_{\overline{S}^{i}} \\ \alpha_{1}^{i}\left(\delta_{1}^{i}\right)^{2} & \alpha_{2}^{i}\left(\delta_{2}^{i}\right)^{2} & \cdots & \alpha_{\overline{S}^{i}}^{i}\left(\delta_{\overline{S}^{i}}^{i}\right)^{2} \\ \vdots & \vdots & \vdots & \vdots \\ \alpha_{1}^{i}\left(\delta_{1}^{i}\right)^{\overline{S}^{i}-1} & \alpha_{2}^{i}\left(\delta_{2}^{i}\right)^{\overline{S}^{i}-1} & \cdots & \alpha_{\overline{S}^{i}}^{i}\left(\delta_{\overline{S}^{i}}^{i}\right)^{\overline{S}^{i}-1} \end{bmatrix}} \times \underbrace{\begin{bmatrix} \overline{\mathbf{w}}_{1}^{i'} \\ \overline{\mathbf{w}}_{3}^{i'} \\ \vdots \\ \overline{\mathbf{w}}_{3}^{i'} \\ \vdots \\ \overline{\mathbf{w}}_{\overline{S}^{i}}^{j'} \end{bmatrix}}_{=\overline{\mathbf{w}}^{i'}}, \qquad i \in \{a, b\}.$$
(74)

Step (ii): The inverse of the τ -period (unperturbed) AD price matrix (74) is

$$\left(\overline{\mathbf{A}}_{\tau}^{i}\right)^{-1} = \left(\overline{\mathbf{W}}^{i'}\right)^{-1} \left(\mathbf{D}^{i}\right)^{-1} = \overline{\mathbf{X}}^{i} \left(\mathbf{D}^{i}\right)^{-1}, \qquad i \in \{a, b\},$$
(75)

where the last equality arises from the normalization $\overline{\mathbf{W}}^{i'}\overline{\mathbf{X}}^{i} = \mathbb{1}_{\overline{S}^{i}\times\overline{S}^{i}}$ of the right and left eigenvector matrices of the one-period AD price matrix $\overline{\mathbf{A}}^{i}$ in the diagonalization (71). To obtain $(\overline{\mathbf{A}}_{\tau}^{i})^{-1}$, we need an explicit expression for the inverse matrix $(\mathbf{D}^{i})^{-1}$, where \mathbf{D}^{i} defined in (74) can be

rewritten as

$$\mathbf{D}^{i} = \underbrace{\begin{bmatrix} 1 & 1 & \cdots & 1 \\ \delta_{1}^{i} & \delta_{2}^{i} & \cdots & \delta_{\overline{S}^{i}}^{i} \\ (\delta_{1}^{i})^{2} & (\delta_{2}^{i})^{2} & \cdots & (\delta_{\overline{S}^{i}}^{i})^{2} \\ \vdots & \vdots & \vdots & \vdots \\ (\delta_{1}^{i})^{\overline{S}^{i}-1} & (\delta_{2}^{i})^{\overline{S}^{i}-1} & \cdots & (\delta_{\overline{S}^{i}}^{i})^{\overline{S}^{i}-1} \end{bmatrix}}_{\equiv \mathbf{G}^{i}} \mathbf{Diag} \left(\alpha_{1}^{i}, \alpha_{2}^{i}, \cdots, \alpha_{\overline{S}^{i}}^{i} \right), \qquad i \in \{a, b\}.$$
(76)

We observe that $\overline{S}^i \times \overline{S}^i$ matrix \mathbf{G}^i defined above is the Vandermonde matrix, whose applications and properties (its determinant and inverse matrix) have been well studied in the literature (see (81) below). In particular, the explicit expression for the inverse of the Vandermonde matrix \mathbf{G}^i (76) reads (see, e.g., Man (2017)),

$$\left(\mathbf{G}^{i}\right)^{-1} = \begin{bmatrix} h_{11} & \dots & h_{1\overline{S}^{i}} \\ \vdots & \ddots & \vdots \\ h_{\overline{S}^{i}_{1}} & \dots & h_{\overline{S}^{i}\overline{S}^{i}} \end{bmatrix}, \qquad i \in \{a, b\},$$
(77)

with

$$h_{mk} = \frac{\left(\delta_m^i\right)^{\overline{S}^i - k} + a_1\left(\delta_m^i\right)^{\overline{S}^i - k - 1} + \dots + a_{\overline{S}^i - k - 1}\left(\delta_m^i\right) + a_{\overline{S}^i - k}}{\prod_{j \neq m} (\delta_m^i - \delta_j^i)}, \quad m, k \in \{1, \dots, \overline{S}^i\},$$

and

$$a_0 = 1;$$
 $a_1 = -\sum_j \delta_j^i;$ $a_2 = \sum_{j \neq m} \delta_j^i \delta_m^i;$...; $a_{\overline{S}^i - 1} = (-1)^{\overline{S}^i - 1} \prod_j \delta_j^i;$

and $a_k = 0$ for all k < 0. Combining (75) and (76) yields the inverse of τ -period (unperturbed) AD price matrix

$$\left(\overline{\mathbf{A}}_{\tau}^{i}\right)^{-1} = \overline{\mathbf{X}}^{i} \mathbf{Diag}\left(\frac{1}{\alpha_{1}^{i}}, \frac{1}{\alpha_{2}^{i}}, \cdots, \frac{1}{\alpha_{\overline{S}^{i}}^{i}}\right) \left(\mathbf{G}^{i}\right)^{-1}, \qquad i \in \{a, b\},$$
(78)

where $(\mathbf{G}^i)^{-1}$ is given explicitly in (77) and α_k^i in (72).

<u>Step (iii)</u>: Recall from (31) that the perturbative component of the recovered time discount factor is $\Delta \delta^i = \frac{(\overline{\mathbf{w}}_1^i)'\overline{\mathbf{B}}^i\overline{\mathbf{x}}_1^i}{(\overline{\mathbf{w}}_1^i)'\overline{\mathbf{x}}_1^i}$. Using (29) and the fact that $\overline{\mathbf{x}}_1^i$ is the first (right) eigenvector of the one-period AD price matrix, $\overline{\mathbf{A}}^i\overline{\mathbf{x}}_1^i = \delta_1^i\overline{\mathbf{x}}_1^i$, the numerator of this perturbative component is

$$\left(\overline{\mathbf{w}}_{1}^{i}\right)'\overline{\mathbf{B}}^{i}\overline{\mathbf{x}}_{1}^{i} = \left(\overline{\mathbf{w}}_{1}^{i}\right)'\left(\overline{\mathbf{A}}_{\tau}^{i}\right)^{-1}\left(\overline{\mathbf{B}}_{\tau+1}^{i} - \overline{\mathbf{B}}_{\tau}^{i}\overline{\mathbf{A}}^{i}\right)\overline{\mathbf{x}}_{1}^{i} = \left(\overline{\mathbf{w}}_{1}^{i}\right)'\left(\overline{\mathbf{A}}_{\tau}^{i}\right)^{-1}\left(\overline{\mathbf{B}}_{\tau+1}^{i} - \delta_{1}^{i}\overline{\mathbf{B}}_{\tau}^{i}\right)\overline{\mathbf{x}}_{1}^{i}$$

Using the expression (78) for the inverse of τ -period AD price matrix and the mutual orthonormality between left and right eigenvectors $\overline{\mathbf{w}}_{j}^{i}$ and $\overline{\mathbf{x}}_{k}^{i}$ of the one-period AD price matrix, we can further simplify the above quantity to

$$(\overline{\mathbf{w}}_{1}^{i})' \overline{\mathbf{B}}^{i} \overline{\mathbf{x}}_{1}^{i} = (\overline{\mathbf{w}}_{1}^{i})' \overline{\mathbf{X}}^{i} \mathbf{Diag} \left(\frac{1}{\alpha_{1}^{i}}, \frac{1}{\alpha_{2}^{i}}, \cdots, \frac{1}{\alpha_{\overline{S}^{i}}^{i}} \right) (\mathbf{G}^{i})^{-1} \left(\overline{\mathbf{B}}_{\tau+1}^{i} - \delta_{1}^{i} \overline{\mathbf{B}}_{\tau}^{i} \right) \overline{\mathbf{x}}_{1}^{i}$$

$$= \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \end{bmatrix} \mathbf{Diag} \left(\frac{1}{\alpha_{1}^{i}}, \frac{1}{\alpha_{2}^{i}}, \cdots, \frac{1}{\alpha_{\overline{S}^{i}}^{i}} \right) (\mathbf{G}^{i})^{-1} \left(\overline{\mathbf{B}}_{\tau+1}^{i} - \delta_{1}^{i} \overline{\mathbf{B}}_{\tau}^{i} \right) \overline{\mathbf{x}}_{1}^{i}$$

$$= \begin{bmatrix} \frac{1}{\alpha_{1}^{i}} & 0 & 0 & \cdots & 0 \end{bmatrix} (\mathbf{G}^{i})^{-1} \left(\overline{\mathbf{B}}_{\tau+1}^{i} - \delta_{1}^{i} \overline{\mathbf{B}}_{\tau}^{i} \right) \overline{\mathbf{x}}_{1}^{i},$$

Substituting in the expressions (68), (69) for matrices $\overline{\mathbf{B}}_{\tau}^{i}$, $\overline{\mathbf{B}}_{\tau+1}^{i}$, $i \in \{a, b\}$, the quantity above becomes

$$\left(\overline{\mathbf{w}}_{1}^{i}\right)'\overline{\mathbf{B}}^{i}\overline{\mathbf{x}}_{1}^{i} = \begin{bmatrix} \frac{1}{\alpha_{1}^{i}} & 0 & 0 & \cdots & 0 \end{bmatrix} \left(\mathbf{G}^{i}\right)^{-1} \begin{bmatrix} 0 & 0 & \cdots & \left(\delta_{1}^{i}\right)^{2}\left(P_{2}-P_{1}\right) \\ 0 & 0 & \cdots & \left(\delta_{1}^{i}\right)^{3}\left(P_{3}-P_{2}\right) \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \left(\delta_{1}^{i}\right)^{\overline{S}^{i}}\left(P_{\overline{S}^{i}}-P_{\overline{S}^{i}-1}\right) \end{bmatrix} \begin{bmatrix} \overline{x}_{11}^{i} \\ \overline{x}_{12}^{i} \\ \vdots \\ \overline{x}_{1\overline{S}^{i}}^{i} \end{bmatrix}, \quad i \in \{a, b\},$$

$$(79)$$

with
$$P_{\tau} = 2p_{t,t+\tau}(1,3) + 3p_{t,t+\tau}(1,4),$$
 (80)

and $\alpha_1^i = \delta_1^i \overline{x}_{11}^i$ (72) and $(\mathbf{G}^i)^{-1}$ given in (77). Substituting the above expression for $(\overline{\mathbf{w}}_1^i)' \overline{\mathbf{B}}^i \overline{\mathbf{x}}_1^i$ into (31) yields the perturbative recovered time discount factors $\delta_1^a(\varepsilon)$ (32) and $\delta_1^b(\varepsilon)$ (33). Note that models *a* and *b* converge to the consistent setting (scheme (25) and Proposition 1) when the perturbation parameter $\varepsilon = 0$. Therefore, the unperturbed dominant eigenvalues coincide $\delta_1^a = \delta_1^b$, so we drop the superscript *i* in formulas (32), (33) the main text <u>A digression on the Vandermonde matrix</u>: To relate the Vandermonde matrix to the recovery setting, we recall a well-known application of this matrix (and its inverse). This application concerns the exact fitting of a (n - 1)-degree polynomial f(x) that passes through n given points $\{(x_0, y_0) \dots (x_{n-1}, y_{n-1})\}$. The fitting "recovers" n unknown coefficients $\{a_0, \dots, a_{n-1}\}$ of polynomial f(x) via an equation system

Vandermonde system:

$$\begin{cases}
 a_0 + a_1 x_0 + a_2 x_0^2 + \ldots + a_{n_1} x_0^{n-1} = y_0, \\
 a_0 + a_1 x_1 + a_2 x_1^2 + \ldots + a_{n_1} x_1^{n-1} = y_1, \\
 \vdots \\
 a_0 + a_1 x_{n-1} + a_2 x_{n-1}^2 + \ldots + a_{n_1} x_{n-1}^{n-1} = y_{n-1},
\end{cases}$$
(81)

A change of notation relates this equation system to our recovery setting (76). Replacing \overline{S}^i by nand δ_k^i by x_k (for $k \in \{1, \ldots, \overline{S}^i\}$) in the matrix \mathbf{G}^i (76), the Vandermonde equation system (81) can be written in the matrix form as $[a_0 \ a_1 \ \ldots \ a_{n-1}] \ \mathbf{G}^i = [y_0 \ y_1 \ \ldots, \ y_{n-1}]$. From this follows a unique solution for the coefficients, $[a_0 \ a_1 \ \ldots \ a_{n-1}] = [y_0 \ y_1 \ \ldots \ y_{n-1}] (\mathbf{G}^i)^{-1}$.

B.2 Recovery Inconsistency Direction: Derivations

This appendix provides technical derivations underlying the intuitive analysis of Section 4.2 on the recovery inconsistency direction of a market model calibrated to the U.S. economy. Recall that the calibrated model has the underlying specification S perceived by the first analyst and a consolidated specification \overline{S} perceived by the second analyst (38),

$$\mathcal{S} = \{1, 2, 3\} \qquad \overline{\mathcal{S}} = \{\overline{1}, \overline{2}\} \text{ with: } \{\overline{1}\} = \{1, 2\}, \ \{\overline{2}\} = \{3\}, \tag{82}$$

where $\{1\}$, $\{2\}$ and $\{3\}$ respectively denotes a rare disaster, a normal, and a boom state (in order of decreasing marginal utility). For a consistent referencing, the three-state calibrated U.S. economy in our analysis below (and in Section 4.2) of the recovery inconsistency direction is specified in Equations (36) (for the transition probabilities, marginal utilities and time preference) and (37) (for the associated one-period AD asset price matrix). As in Section 4.2, we consider with two

cases of current coupled and single states.

<u>Case 1 - Current coupled state</u>: Let the current underlying state be the normal state $\{2\}$. Per the consolidation scheme (82), the second analyst perceives the current state $\{\overline{1}\}$, which is a coupled state. The following lemma presents sufficient conditions for the undershooting (41) of the transition probabilities recovered in our calibration.

Lemma 1 When the underlying market model (associated with the specification S) features (i) $A_{23}A_{32} < A_{22}A_{33}$ and (ii) $A_{22} + A_{33} < A_{23} + A_{32}$, then follows the undershooting of the recovered transition probabilities,

$$\overline{p}_{t,t+1}(\overline{1},\overline{1}) < p_{t,t+1}(2,1) + p_{t,t+1}(2,2), \qquad \overline{p}_{t,t+1}(\overline{1},\overline{1}) < p_{t,t+1}(1,1) + p_{t,t+1}(1,2), \tag{83}$$

As discussed in Section 4.2, the key economic intuition of these sufficient conditions on the inconsistency direction (undershooting) of the recovered transition probabilities is the presence of a rare adverse consumption state (state {1} in the calibrated U.S. economy model.

Proof: We first obtain an explicit expression for the implied one-period AD price matrix $\overline{\mathbf{A}}$ before relate it to the transition probabilities $\{\overline{p}_{\overline{i},\overline{j}}\}$ in the consolidated specification. Note that the calibrated numerical values for the time discount factors δ and $\overline{\delta}$ are very similar (41), so to a good approximation, we take $\delta = \overline{\delta}$ in the analysis. After consolidating long-term AD asset prices in \mathbf{A}_{τ} associated with \mathcal{S} into those in $\overline{\mathbf{A}}_{\tau}$ associated with $\overline{\mathcal{S}}$,² the implied one-period AD price matrix $\overline{\mathbf{A}} = [\overline{\mathbf{A}}_{\tau}]^{-1} \overline{\mathbf{A}}_{\tau+1}$ (39) has the following explicit solution (using 3 first horizons $\tau \in 1, 2, 3$),

$$\overline{\mathbf{A}} = \begin{bmatrix} A_{21} + A_{22} & A_{23} \\ A_{2;21} + A_{2;22} & A_{2;23} \end{bmatrix}^{-1} \begin{bmatrix} A_{2;21} + A_{2;22} & A_{2;23} \\ A_{3;21} + A_{3;22} & A_{3;23} \end{bmatrix} = \frac{1}{\det(\overline{\mathbf{A}}_{\tau})} \times$$

$$\times \begin{bmatrix} A_{2;23}(A_{2;21} + A_{2;22}) - A_{23}(A_{3;21} + A_{3;22}) & A_{2;23}^2 - A_{23}A_{3;23} \\ -(A_{2;21} + A_{2;22})^2 + (A_{21} + A_{22})(A_{3;21} + A_{3;22}) & -A_{2;23}(A_{2;21} + A_{2;22}) + A_{3;23}(A_{21} + A_{22}) \end{bmatrix}.$$

$$(84)$$

²This amounts to summing the first two columns of \mathbf{A}_{τ} , per Footnote 8 and the consolidation scheme (82).

To obtain the first undershooting in (83), we derive a stronger inequality $\overline{p}_{t,t+1}(\overline{1},\overline{1}) < p_{t,t+1}(2,2)$, which is equivalent to $\overline{A}_{\overline{1}\overline{1}} < A_{22}$ by virtue of the pricing equation (3) for one-period AD assets. Using solution (84), we can write this inequality as

$$\frac{A_{2;23}(A_{2;21}+A_{2;22})-A_{23}(A_{3;21}+A_{3;22})}{(A_{21}+A_{22})A_{2;23}-(A_{2;21}+A_{2;22})A_{23}} < A_{22}.$$
(86)

Using again the recursive relationship $\mathbf{A}_{\tau+1} = \mathbf{A}_{\tau} \mathbf{A}$, or explicitly,

$$A_{2;21} = A_{21}A_{11} + A_{22}A_{21} + A_{23}A_{31}$$

$$A_{2;22} = A_{21}A_{12} + A_{22}^2 + A_{23}A_{32}$$

$$A_{2;23} = A_{21}A_{13} + A_{22}A_{23} + A_{23}A_{33}$$

$$A_{3;21} = A_{2;21}A_{11} + A_{2;22}A_{21} + A_{2;23}A_{31}$$

$$A_{3;22} = A_{2;21}A_{12} + A_{2;22}A_{22} + A_{2;23}A_{32}.$$
(87)

we can express the denominator (denoted as D_2) of the fraction on the LHS of (86) as,³

$$D_{2} = A_{21}A_{13}(A_{21} + A_{22}) - A_{21}A_{23}(A_{11} + A_{12}) + A_{22}A_{33}(A_{21} + A_{22}) - A_{23}^{2}(A_{31} + A_{32})$$

$$\approx A_{22}A_{33}(A_{21} + A_{22}) - A_{23}^{2}(A_{31} + A_{32}) \approx A_{22}^{2}A_{33} - A_{23}^{2}A_{32}.$$
(88)

Note that because the normal state $\{2\}$ is associated with a higher marginal utility than the boom state $\{3\}$ (while transition probabilities to these states are similar) in the calibrated model, we have $A_{22} > A_{23}$. Combining this with Lemma 1's condition (i) (that $A_{22}A_{33} > A_{23}A_{32}$), we have a positive denominator $D_2 > 0$ for the fraction on the LHS of (86), transforming (86) into

$$A_{2;23}(A_{2;21} + A_{2;22}) - A_{23}(A_{3;21} + A_{3;22}) < A_{22}\left[(A_{21} + A_{22})A_{2;23} - (A_{2;21} + A_{2;22})A_{23}\right] + A_{2;22}(A_{2;21} + A_{2;22})A_{23} = 0$$

³The first approximation in (88) arises from the argument below Equation (37). Namely, the exceedingly rare incidence of a disaster state {1} in the underlying model sufficiently negates the severity of that state, resulting in a sufficiently low underlying disaster state price (and prices of AD assets paying off in state {1}), we have $A_{21}A_{13}(A_{21} + A_{22})$ is dominated by $A_{22}A_{33}(A_{21} + A_{22})$ (because the first group is quadratic in A_{i1} , and the second is linear in A_{i1}), and similarly $A_{21}A_{23}(A_{11} + A_{12})$ is dominated by $A_{23}^2(A_{31} + A_{32})$, giving rise to the first approximation. The second approximation in (88) by a similar argument, which implies the dominance of A_{21} by A_{22} , and A_{31} by A_{32} .

Using relationships (87) to substitute out the longer-term AD prices $\{A_{2;ij}\}, \{A_{3;ij}\}$, and rearranging terms, the above inequality becomes

$$[A_{22}(A_{21}A_{11} + A_{23}A_{31}) - A_{21}(A_{21}A_{12} + A_{23}A_{32})]A_{23} + [A_{21}(A_{21}A_{13} + A_{23}A_{33}) - A_{23}(A_{21}A_{11} + A_{23}A_{31})](A_{11} + A_{12}) < 0.$$
(89)

We simplify the above inequality by concentrating on the dominant terms (again because $\{1\}$ is a rare disaster state),

$$[A_{22}A_{23}A_{31} - A_{21}A_{23}A_{32}]A_{23} + [A_{21}A_{23}A_{33} - A_{23}A_{23}A_{31}](A_{11} + A_{12}) < 0$$
(90)

Note that $A_{22}A_{31} < A_{21}A_{32}$ and $(A_{11} + A_{12}) < A_{23}$,⁴ the LHS of the inequality (90) is smaller than $[A_{22}A_{23}A_{31} - A_{21}A_{23}A_{32} + A_{21}A_{23}A_{33} - A_{23}A_{23}A_{31}]A_{23}$. Then a sufficient condition for (90) is

$$[A_{22}A_{23}A_{31} - A_{21}A_{23}A_{32} + A_{21}A_{23}A_{33} - A_{23}A_{23}A_{31}]A_{23} < 0$$

$$\Leftrightarrow [A_{22}A_{31} - A_{21}A_{32} + A_{21}A_{33} - A_{23}A_{31}] = A_{31}(A_{22} - A_{23}) - A_{21}(A_{32} - A_{33}) < 0.$$
(91)

Because the chance of reaching the disaster state $\{1\}$ from the boom state $\{3\}$ is lower than from from the normal state $\{2\}$, i.e., $p_{31} < p_{21}$, whose effect also dominates the difference between the marginal utilities in the boom and normal states in the calibrated model, we have $A_{31} < A_{21}$. Furthermore, by a similar argument, we have $(A_{22} - A_{23}) > 0$ and $(A_{32} - A_{33}) > 0$. Therefore a sufficient condition for the inequality (91) is $(A_{32} - A_{33}) > (A_{22} - A_{23})$, which is the condition (ii) in Lemma 1. Arranging these implication steps in the reverse order yields the proof that conditions (i) and (ii) in Lemma 1 imply the first undershooting relationship in (41) (or (83)) of the recovered probabilities in the subjective specification. The second undershooting relationship follows from a similar derivation thread, which centers around the presence of the rare disaster state $\{1\}$ in the calibrated economy as discussed in Section 4.2

⁴ For the first inequality, we note that $A_{22}A_{31} = \delta^2 p_{22}p_{31}\frac{M_1}{M_3}$, $A_{21}A_{32} = \delta^2 p_{21}p_{32}\frac{M_1}{M_3}$, while numerically $p_{32} \approx p_{22}$ and $p_{31} < p_{21}$ (because the likelihood of getting to the disaster state {1} next period from the boom state {3} is lower than from the normal state {2} in the calibrated model to the U.S. economy), implying $A_{22}A_{31} < A_{21}A_{32}$. For the second inequality, we note that $(A_{11} + A_{12}) = \delta p_{11} + \delta p_{12}\frac{M_2}{M_1}$, $A_{23} = \delta p_{23}\frac{M_3}{M_2}$, $p_{11} \ll p_{23}$ (because the underlying disaster state {1} is exceedingly rare) and $\frac{M_2}{M_1} \ll \frac{M_3}{M_2}$ (because the disaster state state's marginal utility M_1 is high), implying $(A_{11} + A_{12}) < A_{23}$.
<u>Case 2 - Current single state</u>: Now let the current underlying state be the boom state $\{3\}$. Per the consolidation scheme (82), the second analyst perceives the current state $\{\overline{2}\}$, which is a single state. The following lemma presents sufficient conditions for the overshooting (49) of the transition probabilities recovered in our calibration.

Lemma 2 When the underlying market model (associated with the specification S) features $A_{23}A_{32} < A_{22}A_{33}$, then follows the undershooting of the recovered transition probabilities $\overline{p}_{t,t+1}(\overline{2},\overline{2}) > p_{t,t+1}(3,3)$ (49).

Note that the this lemma's sufficient condition is weaker than those underlying the Lemma 1, implying that a single economic setting that explains the undershooting (83) of the transition probabilities recovered with the current coupled state $\{2\}$ also explains the overshooting of the transition probability recovered with the current single state $\{3\}$ in Lemma 2.

Proof: We also work out an explicit expression for the implied one-period AD price matrix $\overline{\mathbf{A}}$. But as the current state is {3}, this expression (92) for $\overline{\mathbf{A}}$ differs from (84) (which is for the current state {2}). Specifically, after consolidating long-term AD asset prices in \mathbf{A}_{τ} in \mathcal{S} into those in $\overline{\mathbf{A}}_{\tau}$ in $\overline{\mathcal{S}}$, the recursive formula $\overline{\mathbf{A}} = [\overline{\mathbf{A}}_{\tau}]^{-1} \overline{\mathbf{A}}_{\tau+1}$ (39) yields an explicit solution for the implied one-period AD price matrix in the consolidated specification,

$$\overline{\mathbf{A}} = \begin{bmatrix} A_{31} + A_{32} & A_{33} \\ A_{2;31} + A_{2;32} & A_{2;33} \end{bmatrix}^{-1} \begin{bmatrix} A_{2;31} + A_{2;32} & A_{2;33} \\ A_{3;31} + A_{3;32} & A_{3;33} \end{bmatrix}$$
(92)

$$= \frac{1}{det(\overline{\mathbf{A}}_{\tau})} \begin{bmatrix} A_{2;33}(A_{2;31} + A_{2;32}) - A_{33}(A_{3;31} + A_{3;32}) & A_{2;33}^2 - A_{33}A_{3;33} \\ -(A_{2;31} + A_{2;32})^2 + (A_{31} + A_{32})(A_{3;31} + A_{3;32}) & -A_{2;33}(A_{2;31} + A_{2;32}) + A_{3;33}(A_{31} + A_{32}). \end{bmatrix}$$

To a good approximation, we also take $\delta = \overline{\delta}$ in the analysis as before. Note that the overshoot relationship $\overline{p}_{t,t+1}(\overline{2},\overline{2}) > p_{t,t+1}(3,3)$ in Lemma 2 is equivalent to $\overline{A}_{\overline{2}\overline{2}} > A_{33}$ by virtue of the pricing equation (3) for one-period AD assets. Using solution (92), we can write this inequality as

$$\frac{-A_{2;33}(A_{2;31}+A_{2;32})+A_{3;33}(A_{31}+A_{32})}{(A_{31}+A_{32})A_{2;33}-(A_{2;31}+A_{2;32})A_{33}} > A_{33}.$$
(93)

Using again the recursive relationship $\mathbf{A}_{\tau+1} = \mathbf{A}_{\tau} \mathbf{A}$, or explicitly,

$$A_{2;31} = A_{31}A_{11} + A_{32}A_{21} + A_{33}A_{31}$$

$$A_{2;32} = A_{31}A_{12} + A_{32}A_{22} + A_{33}A_{32}$$

$$A_{2;33} = A_{31}A_{13} + A_{32}A_{23} + A_{33}^{2}$$

$$A_{3;31} = A_{2;31}A_{11} + A_{2;32}A_{21} + A_{2;33}A_{31}$$

$$A_{3;32} = A_{2;31}A_{12} + A_{2;32}A_{22} + A_{2;33}A_{32}$$

$$A_{3;33} = A_{2;31}A_{13} + A_{2;32}A_{23} + A_{2;33}A_{33}.$$
(94)

we can express the denominator (denoted as D_3) of the fraction on the LHS of (93) in terms of one-period AD asset prices as,

$$D_3 = (A_{31} + A_{32})A_{13}A_{31} + (A_{31} + A_{32})A_{23}A_{32} - (A_{11} + A_{12})A_{31}A_{33} - (A_{21} + A_{22})A_{32}A_{33}.$$
 (95)

First, we show that this denominator is negative, $D_3 < 0$. To this end, note that among 4 terms of D_3 , the first term is dominated by the second term, the third is dominated by the fourth.⁵ As a result, we can approximately simplify D_3 as

$$D_3 = A_{32} \left[(A_{31} + A_{32})A_{23} - (A_{21} + A_{22})A_{33} \right].$$

Because $A_{31}A_{23} < A_{21}A_{33}$,⁶ we have $D_3 < A_{32}[A_{32}A_{23} - A_{22}A_{33}]$. By virtue of the condition $A_{32}A_{23} < A_{22}A_{33}$ in Lemma 2, we have $D_3 < 0$.

Next, since the denominator D_3 of the fraction on the LHS of inequality (93) is negative, (93)

⁵That is, $|(A_{31} + A_{32})A_{13}A_{31}| \ll |(A_{31} + A_{32})A_{23}A_{32}|$, and $|(A_{11} + A_{12})A_{31}A_{33}| \ll |(A_{21} + A_{22})A_{32}A_{33}|$. This is because as {1} is a rare disaster state, the transition probabilities p_{i1} from any underlying state {i} to the underlying disaster state {1} next period is exceedingly small. As a result, the prices { A_{i1} } of insurances against state {1} are dominated by the prices { A_{j2}, A_{h2} } of insurances against non-disaster states {2,3}.

⁶Note that $A_{31} < A_{21}$ (per the discussion below (91)). Similarly, $A_{23} < A_{33}$ (because $p_{23} \approx p_{33}$ and their difference is dominated by the difference between the marginal utilities in the boom and normal states, $M_3 < M_2$).

is equivalent to

$$-A_{2;33}(A_{2;31} + A_{2;32}) + A_{3;33}(A_{31} + A_{32}) < A_{33}\left[(A_{31} + A_{32})A_{2;33} - (A_{2;31} + A_{2;32})A_{33}\right].$$

Using (94) to substitute out the longer-term AD asset prices $\{A_{2;ij}, A_{3;ij}\}$ and rearrange terms, the above equality becomes $(A_{23} - A_{13}) [A_{31}A_{32}(A_{22} - A_{11}) + A_{31}^2A_{12} - A_{32}^2A_{21}] < 0$. Since $A_{23} - A_{13} > 0$, in turn this is equivalent to,⁷

$$A_{31}A_{32}(A_{22}-A_{11}) + A_{31}^2A_{12} - A_{32}^2A_{21} = A_{32}(A_{31}A_{22} - A_{32}A_{21}) + A_{31}(A_{31}A_{12} - A_{32}A_{11}) < 0, (96)$$

Note that the second group of terms $A_{31}(A_{31}A_{12} - A_{32}A_{11})$ is dominated by the first $A_{32}(A_{31}A_{22} - A_{32}A_{21})$ (because the first group is linear and the second in quadratic in prices $\{A_{i1}\}$, which are very small because the underlying state $\{1\}$ is exceedingly rare, see Footnote 3). We therefore can rewrite (96) as $A_{32}(A_{31}A_{22} - A_{32}A_{21}) < 0$. This is true because $A_{31}A_{22} - A_{32}A_{21}$ as explained below Equation (90) and Footnote 4. Arranging these implication steps in the reverse order yields a proof of Lemma 2

B.3 Recovery Inconsistency Direction: Calibration Details

Data source: We use three data sets: US household real consumption expenditure data from OECD, S&P 500 price data from Yahoo Finance, and 3-month Treasury Bill rate data from FRED. All three data sets are at quarterly frequency from 1985Q1 to 2022Q4.

Model specification: We calibrate to a 3-state, 2-period (t and t+1) consumption-based CAPM (CCAPM) model, where the utility function is $\frac{c^{1-\gamma}}{1-\gamma}$ and $c \in \{c_1, c_2, c_3\}$. Suppose the current state at t is the normal state (state 2). The pricing kernel at time t is $\Lambda_t = \delta \left(\frac{c_{t+1}}{c_t}\right)^{-\gamma}$, where $c_t = c_2$ and $c_{t+1} \in \{c_1, c_2, c_3\}$.

Since we focus on an ergodic Markov chain (i.e., every state is reachable over a finite period of time), in the long run, the time-series average and the cross-sectional average coincide. As our

⁷Note that $A_{23} - A_{13} = \delta \left[p_{23} \frac{M_3}{M_2} - p_{13} \frac{M_3}{M_1} \right]$. Because the disaster state {1} has significantly higher marginal utility $(M_1 > M_2)$ while the transition probabilities to the boom state {3} next period are similar $p_{23} \approx p_{13}$, we have $A_{23} - A_{13} > 0$.

sample period is sufficiently long, it is reasonable to analyze the moments under the stationary distribution. Denote the stationary probabilities as $p_{ss} = [p_1, p_2, p_3]'$, which is the left eigenvector (normalized to have sum equal to 1) of **P** corresponding to the eigenvalue equal to 1.

The risk-free rate r_{t+1}^f from t to t+1 must satisfy the following Euler equation

$$\frac{1}{1+r_{t+1}^f} = \mathbb{E}_t \left[\Lambda_{t+1} \right],\tag{97}$$

where the expectation is taken over p_{ss} , and the risk-free rate is known at t.

To price the aggregate stock market (a risky asset), note that the stock market price is defined to be the value of discounted dividends. In the Lucas tree model, the equilibrium consumption is equal to the dividend of the aggregate stock market in each period. Therefore, we have $S_t = \mathbb{E}_t [\Lambda_{t+1}c_{t+1}]$, or in terms of stock return, we have $1 = \mathbb{E}_t [\Lambda_{t+1}(1+r_{t+1}^s)]$, where $r_{t+1}^s = \frac{c_{t+1}}{S_t} - 1$ is the return on the stock from t to t + 1. The above equation then becomes the following CCAPM formula

$$\mathbb{E}_t[r_{t+1}^s] - r_{t+1}^f = -(1 + r_{t+1}^f) Cov_t \left(\Lambda_{t+1}, r_{t+1}^s\right).$$
(98)

Parameter estimation: Our goal is to find a 3×3 probability matrix $\mathbf{P} = [p_{ij}]$ whose stationary distribution will match the moments below. Since the calibration is left with several degrees of freedom, we exogenously specify the transition probabilities to model three stylized but basic states of the economy, namely a rare adverse (disaster, low consumption) state $\{1\}$, a normal (moderate-consumption) state $\{2\}$, and a boom (high-consumption) state $\{3\}$. We first make a exogenous (random-generated) choice for the triplet transition probabilities $\{p_{i1}\}, \{i\} \in \{1, 2, 3\}$, with upper bound of 0.05 each, and constrained by $p_{11} > p_{21} > p_{31}$. These exogenous constraints aim to model and stylize state $\{1\}$ as a rare disaster state, that is more likely to reach when the current state is (i) a disaster state, than (ii) a normal state, than (iii) a boom state. We then employ $\{p_{i1}\}$ as inputs for the calibration. Since each row of \mathbf{P} sums to one, we are left with 3 probabilities to estimate, i.e., p_{12}, p_{22} , and p_{32} .

We will match the following moments to the data.

Average consumption growth:
$$\mu_c = \mathbb{E}_t \left[\frac{c_{t+1}}{c_t} - 1 \right]$$
 (99)

Consumption growth volatility:

$$\sigma_c^2 = \mathbb{E}_t \left[\left(\frac{c_{t+1}}{c_t} - 1 - \mu_c \right)^2 \right]$$
(100)

Risk premium of stock:
$$rx_t^s = -(1 + r_{t+1}^f)Cov_t\left(\Lambda_{t+1}, r_{t+1}^s\right)$$
(101)

Stock return volatility:

$$\sigma_s^2 = \mathbb{E}_t \left[\left(\frac{c_{t+1}}{\mathbb{E}_t \left[\Lambda_{t+1} c_{t+1} \right]} - 1 - (r x_t^s + r_{t+1}^f) \right)^2 \right] \quad (102)$$

Risk-free rate:
$$\frac{1}{1+r_{t+1}^f} = \mathbb{E}_t \left[\Lambda_{t+1} \right].$$
(103)

Note that in our estimation, all quantities on the RHS of the above equations are taken from the 3-state model while all quantities on the LHS are from the actual data.

Define g to be a vector containing the 5 moment conditions

$$g = \begin{bmatrix} \mathbb{E}_{t} \left[\frac{c_{t+1}}{c_{t}} - 1 \right] - \mu_{c} \\ \mathbb{E}_{t} \left[\left(\frac{c_{t+1}}{c_{t}} - 1 - \mu_{c} \right)^{2} \right] - \sigma_{c}^{2} \\ -(1 + r_{t+1}^{f}) Cov_{t} \left(\Lambda_{t+1}, r_{t+1}^{s} \right) - rx_{t}^{s} \\ \mathbb{E}_{t} \left[\left(\frac{c_{t+1}}{\mathbb{E}_{t}[\lambda_{t+1}c_{t+1}]} - 1 - (rx_{t}^{s} + r_{t+1}^{f}) \right)^{2} \right] - \sigma_{s}^{2} \\ \mathbb{E}_{t} \left[\Lambda_{t+1} \right] - \frac{1}{1 + r_{t+1}^{f}} \end{bmatrix},$$

where the expectation is over the stationary probabilities p_{ss} . Given the weight matrix A, we choose p_{12}, p_{22}, p_{32} to minimize J = g'Ag subject to the constraint $0 \le p_{12}, p_{22}, p_{32} \le 1$ and $p_{13} = 1 - p_{11} - p_{12}, p_{23} = 1 - p_{21} - p_{22}, p_{33} = 1 - p_{31} - p_{32}$. The standard two-stage GMM estimation results are given in Section 4.1.

C Extensions to Basic Recovery Framework

We discuss the generalized recovery and best-fit recovery approaches and the associated consistency issues respectively in Sections C.1 and C.2.

C.1 Generalized Recovery

Recall that the basic recovery approach assumes time-homogenous state space dynamics (Assumption A2) to imply the entire one-period AD price matrix, which is needed to recover the transition probability between any two states via the Recovery Theorem. Not only the determination of this matrix is intricate, upholding the time-homogeneity assumption also rules out interesting the state space dynamics. To address these limitations, Jensen et al. (2019)'s (or JLP (2019) hereafter) relax the time-homogeneity assumption and establish a generalized recovery approach. This appendix briefly describes the generalized recovery and discusses the associated consistency issue.

Generalized recovery's setup: While relaxing the time-homogeneity requirement for the state transition dynamics, the generalized recovery aims to recover the transition probabilities starting only from the actual current state of the economy (say, state $\{i\}$ at the current time t = 0). The pricing of the τ -period AD assets $\{A_{\tau;1j}\}$, for $\{\{j\} \in S\}$, implies a relation between AD prices and the corresponding transition probabilities,⁸

$$A_{\tau;1j}\frac{1}{M_j} = \delta^{\tau} p_{0,\tau}(1,j)\frac{1}{M_1}, \quad \forall \{j\} \in \mathcal{S}, \tau \in \{1,\dots,T\}.$$
(104)

Consequently, the summation over all final states $\{j\}$'s, together with the condition $\sum_{\{j\}} p_{0,\tau}(1,j) = 1$, generates the key equation system for the generalized recovery,

$$\sum_{\{j\}\in\mathcal{S}} A_{\tau;1j} \frac{M_1}{M_j} = \delta^{\tau}, \quad \text{or} \quad A_{\tau;11} + \sum_{\{j\}=2}^{\{S\}} A_{\tau;1j} \frac{M_1}{M_j} = \delta^{\tau}, \quad \forall \tau \in \{1, \dots, T\},$$
(105)

which determines the time discount factor δ and risk preferences $\left\{\frac{M_j}{M_1}\right\}$, $\{j\} \in \{1, \ldots, S\}$. Generalizing (3), the transition probability from the current state to any state $\{j\}$ at time τ then follows from (104)

$$p_{0,\tau}(1,j) = \delta^{-\tau} A_{\tau;1j} \frac{M_1}{M_j}, \quad \{j\} \in \{1,\dots,S\}, \quad \tau \in \{1,\dots,T\}.$$
 (106)

Note that in contrast to Ross's recovery, the generalized recovery system is non-linear and employs

⁸To arrive at this relationship, note that the pricing of τ -period AD assets is, $A_{\tau;1j} = E_0 \left[\delta^{\tau} \frac{M_{s_{\tau}}}{M_1} \mathbb{1}_j(s_{\tau}) \right] = p_{0,\tau}(1,j)\delta^{\tau} \frac{M_j}{M_1}$, where the indicator function $\mathbb{1}_j(s_{\tau})$ denotes the payoff of AD asset $A_{\tau;1j}$.

price data of all T available horizons while being limited to the initial state being the actual current state {1}. The generalized recovery centers on the counting arguments concerning the system (105) of S unknowns $\left\{\delta, \frac{M_1}{M_2}, \ldots, \frac{M_1}{M_S}\right\}$ and T nonlinear equations (one for each value of τ in $\{1, \ldots, T\}$).⁹ When the number of unknowns is greater than or equal to that of equations, $S \geq T$, there are multiple solutions of this nonlinear equation system in general, ruling out an unambiguous (unique) recovery of the transition probabilities, time and risk preferences. When S < T, the system does not have a solution in general, also ruling out a successful recovery. However, JLP (2019) observe that when AD price data arise from a no-arbitrage asset pricing model consistent with a time-separable preference (Assumption A1), the system (105) has a unique solution,¹⁰ which conceptually constitutes the generalized recovery.

Consistency issue in generalized recovery: Intuitively, compared to the Ross's original recovery equation system (which is linear and recursive), the generalized recovery system (106) is non-linear. This non-linearity would exacerbate any inconsistencies originated with a subjective state space specification that does not satisfy the necessary and sufficient condition for consistency (Proposition 1). Specifically, consider two different analysts implementing the generalized recovery process respectively using their subjective input specifications $S = \{1, \ldots, S\}$ (original) and $\overline{S} = \{\overline{1}, \ldots, \overline{S}\}$ (consolidated). We examine whether their recovery results are reconcilable, i.e. the consistency issue, in two exhaustive scenarios, namely, the current state being a single or a coupled state.

<u>Case 1 - Current single state</u>: Let the first K consolidated states be single states and the remaining consolidated state be a coupled state (with $K = \overline{S} - 1$), namely, $\{\overline{1}\} = \{1\}, \ldots, \{\overline{K}\} = \{K\}, \{\overline{S}\} = \{K + 1, \ldots, S\}$. Let the current state be the first single state $\{1\} = \{\overline{1}\}$. While the first (original) analyst solves the generalized recovery system (105) associated with S, the second

⁹Because SDF can only be determined up to a multiplicative constant, the recovery process concerns only the ratios of marginal utilities $\left\{\frac{M_1}{M_2}, \ldots, \frac{M_1}{M_S}\right\}$. Equivalently, we can normalize the marginal utility in the first state to be one, $M_1 = 1$, without loss of generality.

¹⁰Intuitively, in this case, price data are redundant and also consistent because they arise from the same underlying market model. As a result, data redundancy does not lead to inconsistencies in the solution of the system (105), and the generalized recovery works.

(consolidated) analyst solves a similar system associated with $\overline{\mathcal{S}}$,

$$\sum_{\{\overline{j}\}=\{\overline{1}\}}^{\{K\}} \overline{A}_{\tau;\overline{1}\,\overline{j}} \, \frac{\overline{M}_{\overline{1}}}{\overline{M}_{\overline{j}}} \, + \, \overline{A}_{\tau;\overline{1}\,\overline{S}} \, \frac{\overline{M}_{\overline{1}}}{\overline{M}_{\overline{S}}} = \overline{\delta}^{\tau}, \quad \forall \tau \in \{1,\ldots,T\}.$$
(107)

where $A_{\tau;1j}$ and $\overline{A}_{\tau;\overline{1}\,\overline{j}}$ denote the τ -period observable AD asset prices initiated on the current state $\{1\} = \{\overline{1}\}$ that pay respectively on states $\{j\}$ and $\{\overline{j}\}$ of the original and consolidated specifications. These AD asset prices are related by the no-arbitrage principle,

$$\overline{A}_{\tau;\overline{1}\,\overline{j}} = A_{\tau;1j}, \,\forall\{\overline{j}\} = \{j\} \in \{1,\dots,K\}; \qquad \overline{A}_{\tau;\overline{1}\,\overline{S}} = \sum_{\{j\}=\{K+1\}}^{\{S\}} A_{\tau;1j}; \qquad \forall \tau \in \{1,\dots,T\}, \quad (108)$$

Given two sets of recovery results in S and \overline{S} , the following consistency conditions set the criteria to assure the compatibility of recovery results obtained by different analysts

time discount factor: $\overline{\delta} = \delta$, (109) transition probability: $\begin{cases} \overline{p}_{0,\tau}(\overline{1},\overline{j}) = p_{0,\tau}(1,j), & \forall \{\overline{j}\} = \{j\} \in \{1,\dots,K\}, \\ \overline{p}_{0,\tau}(\overline{1},\overline{S}) = \sum_{\{\ell\}=\{K+1\}}^{\{S\}} p_{0,\tau}(1,\ell), & \forall \tau \in \{1,\dots,T\}, \end{cases}$ (110)

marginal utilities :
$$\begin{cases} \frac{\overline{M}_{\overline{j}}}{\overline{M}_{\overline{1}}} = \frac{M_{j}}{M_{1}}, & \forall \{\overline{j}\} = \{j\} \in \{1, \dots, K\}, \\ \frac{\overline{M}_{\overline{S}}}{\overline{M}_{\overline{1}}} = \sum_{\{\ell\} = \{K+1\}}^{\{S\}} \frac{p_{0,\tau}(1,\ell)}{\sum_{\{h\} = \{K+1\}}^{\{S\}} p_{0,\tau}(1,h)} \frac{M_{\ell}}{M_{1}}, & \forall \tau \in \{1, \dots, T\}. \end{cases}$$
(111)

When at least some of the consistency conditions do not hold, the consistency issue in the generalized recovery arises. Intuitively, the recovery results in an input specification (\mathcal{S} or $\overline{\mathcal{S}}$) are obtained from solving the respective nonlinear equation system ((105) or (107)). These equation systems for different specifications are only loosely related by the no-arbitrage relationships (108) between the AD asset prices { $A_{\tau;1j}$ } and { $\overline{A}_{\tau;\overline{1}\,\overline{j}}$ } observed by analysts. Whereas, the solutions of these loosely related nonlinear equation systems need to satisfy a set of strict consistency conditions (109)-(111), indicating the challenge of a consistent generalized recovery. Quantitatively, to demonstrate this challenge, we first assume that the original \mathcal{S} (employed by the first analyst) is the underlying state space specification of the market model. In this premise, the generalized recovery works for S and the first analyst's equation system (105) recovers unambiguously the characteristics (transition probabilities starting from the current state, and time and risk preferences) of the underlying market model (JLP (2019)).¹¹ Perceiving a different subjective specification \overline{S} , the second analyst employs the (consolidated) AD asset prices $\{\overline{A}_{\tau;\overline{1}\,\overline{j}}\}$ (108) to formulate and solve the recovery system (107). In the interest of a recovery consistency analysis, we assume that the generalized recovery works under the (consolidated) specification \overline{S} .¹² As the (in)consistency between recovery results in S and \overline{S} is reflected in the (in)compatibility between the recovery equation systems (105) and (107), we first employ the AD price relation (108) and the required conditions (109), (111) to express the consolidated recovery system (107) in terms of original prices $\{A_{\tau;1j}\}$,

$$\sum_{\{j\}=\{1\}}^{\{K\}} A_{\tau;1j} \, \frac{M_1}{M_j} + \sum_{\{j\}=\{K+1\}}^{\{S\}} A_{\tau;1j} \, \frac{\overline{M}_{\overline{1}}}{\overline{M}_{\overline{S}}} = \delta^{\tau}, \quad \forall \tau \in \{1, \dots, T\}.$$
(112)

We observe that system (112) is compatible with the original recovery system (105) only when $\sum_{\{j\}=\{K+1\}}^{\{S\}} A_{\tau;1j} \frac{M_1}{M_j} = \sum_{\{j\}=\{K+1\}}^{\{S\}} A_{\tau;1j} \frac{\overline{M_1}}{\overline{M_S}}$, for all $\tau \in \{1, \ldots, T\}$. For each τ , this equation presents a compatibility condition for systems (105) and (107). In the current analysis setting (with S being the true underlying specification), the AD prices $\{A_{\tau;1j}\}$ and underlying marginal utilities $\{M_j\}$ are the (consistent) given inputs, the consolidated marginal utilities $\{\overline{M_j}\}$ are implied (recovered). Hence, for the compatibility condition to hold for all $\tau \in \{1, \ldots, T\}$, the only possibility is that the underlying marginal utilities for all single states be identical: $M_j = \kappa, \forall \{j\} \in \{\{K + 1\}, \ldots, \{S\}\}$. In this premise, these underlying marginal utilities are equal to the recovered marginal utility $\kappa = \overline{M_S}$, i.e., the recover results are consistent. We note that this condition on the underlying marginal utilities is identical to that concerning the Ross's recovery (Proposition 1).

<u>Case 2 - Current coupled state</u>: Now let the first consolidated state $\overline{1}$ be a coupled state and and the remaining $\overline{S} - 1$ consolidated states be single states, namely, $\{\overline{1}\} = \{1, \dots, K\}, \{\overline{K+1}\} = \{K+1\},$

¹¹Recall that the key point of generalized recovery is that, when it works, the number of data horizons can be larger than the number of states, T > S, as discussed below (106).

¹²With regard to the second analyst's perspective (\overline{S}) , there are only two possibilities; either (i) the generalized recovery works (i.e., system (107) has a unique solution), or (ii) the generalized recovery does not work (i.e., system (107) has none or multiple solutions). We rule out the possibility (ii) in our analysis because in this case, the recovery works for the first but not the second analyst, making their recovery results outright incompatible.

..., $\{\overline{S}\} = \{S\}$. Let the current underlying state be the first state $\{1\}$ of the original specification S. Therefore, the second analyst perceives the coupled state $\{\overline{1}\}$ as the current state. The observed τ -period AD asset prices $\{A_{\tau;1j}\}$ and $\{\overline{A}_{\tau;\overline{1}\overline{j}}\}$ are interpreted by the two analysts according to their perceived specifications, and similar to (108), are related by the no-arbitrage principle

$$\overline{A}_{\tau;\overline{1}\,\overline{1}} = \sum_{\{j\}=\{1\}}^{\{K\}} A_{\tau;1j}; \qquad \overline{A}_{\tau;\overline{1}\,\overline{j}} = A_{\tau;1j}, \ \{\overline{j}\} = \{j\} \in \{K+1,\dots,S\}; \quad \forall \tau \in \{1,\dots,T\}.$$
(113)

The first analyst solves the original generalized recovery system (105), and the second solves a similar system associated with the consolidated specification \overline{S} ,

$$\overline{A}_{\tau;\overline{1}\,\overline{1}} + \sum_{\overline{j}=\overline{K+1}}^{\overline{S}} \overline{A}_{\tau;\overline{1}\,\overline{j}} \ \frac{\overline{M}_{\overline{1}}}{\overline{M}_{\overline{j}}} = \overline{\delta}^{\tau}, \quad \forall \tau \in \{1,\dots,T\}.$$
(114)

(116)

Similar to (109)-(117), the consistency conditions for the two analysts' recovery results are

$$\text{time discount factor}: \quad \overline{\delta} = \delta, \tag{115}$$

$$\text{transition probabilities}: \begin{cases} \overline{p}_{0,\tau}(\overline{1},\overline{1}) = \sum_{\{\ell\}=\{1\}}^{\{K\}} p_{0,\tau}(1,\ell) \frac{M_{\ell}}{M_{1}}, & \forall \tau \in \{1,\ldots,T\}, \\ \\ \overline{p}_{0,\tau}(\overline{1},\overline{j}) = p_{0,\tau}(1,j) \frac{1 - \sum_{\{\ell\}=\{1\}}^{\{K\}} p_{0,\tau}(1,\ell) \frac{M_{\ell}}{M_{1}}}{1 - \sum_{\{\ell\}=\{1\}}^{\{K\}} p_{0,\tau}(1,\ell)}, & \forall \{\overline{j}\} = \{j\} \in \{K+1,\ldots,S\} \end{cases}$$

$$\text{marginal utilities}: \quad \frac{\overline{M}_{\overline{j}}}{\overline{M}_{\overline{1}}} = \frac{M_{j}}{M_{1}} \underbrace{\frac{1 - \sum_{\{\ell\} = \{1\}}^{\{K\}} p_{0,\tau}(1,\ell)}{1 - \sum_{\{\ell\} = \{1\}}^{\{K\}} p_{0,\tau}(1,\ell) \frac{M_{\ell}}{M_{1}}}}_{\equiv H}, \qquad \qquad \forall \{\overline{j}\} = \{j\} \in \{K+1,\ldots,S\},$$

$$\forall \tau \in \{1,\ldots,T\}.$$

$$(117)$$

Intuitively, because the non-linear recovery systems (105) in S and (114) in \overline{S} are only loosely related (via the relationships (113) between τ -period AD asset prices observed by analysts), their solutions in general do not satisfy the strict consistency conditions (115)-(117), indicating the consistency issue for the generalized recovery. Quantitatively, the consistency issue can be seen as a conflict when we assume that the generalized recovery works for both specifications and uphold conditions (115)-(117) as required for the consistency. To this end, we employ the AD price relationships (113) and consistency conditions (115) and (117) to express the consolidated recovery system (114) in terms of original prices $\{A_{\tau;1j}\}$,

$$\sum_{\{j\}=\{1\}}^{\{K\}} A_{\tau;1j} + \sum_{\{j\}=\{K+1\}}^{\{S\}} A_{\tau;1j} \frac{M_1}{M_j} H = \delta^{\tau}, \quad \forall \tau \in \{1, \dots, T\},$$
(118)

where the state-independent parameter $H = \frac{1 - \sum_{\{j\}=\{1\}}^{\{K\}} p_{0,\tau}(1,j)}{1 - \sum_{\{j\}=\{1\}}^{\{K\}} p_{0,\tau}(1,j) \frac{M_j}{M_1}}$ is defined in (117). We observe that system (118) is compatible with the original recovery system (105) only when $\frac{M_1}{M_j} = 1$, for all $\{j\} \in \{\overline{1}\} = \{1, \ldots, K\}$.¹³ We note that this condition on the underlying marginal utilities is also identical to that concerning the Ross's recovery (Proposition 1).

To summarize the analysis for both cases (single and coupled current state), the generalized recovery results obtained under the two specifications $S \supset \overline{S}$ are mutually consistent if and only if all single states $\{j\} \in S$ that correspond to a coupled state $\{\overline{j}\} \in \overline{S}$ are indistinguishable from the coarser specification \overline{S} 's perspective. This condition mirrors Proposition 1 for Ross's recovery and indicates an elusive nature of a consistent generalized recovery implementation.

C.2 Best-Fit Recovery

Given a subjective (input) specification S of S states, the basic recovery approach employs price data of just enough S + 1 maturities τ of long-term AD assets in the recursive equation system $\mathbf{A}_{\tau+1} = \mathbf{A}_{\tau}\mathbf{A}$ (4) to imply exactly the one-period $S \times S$ AD price matrix \mathbf{A} . Another analyst associated with a different subjective specification \overline{S} of $\overline{S} < S$ states just needs to employ less price data (of $\overline{S} + 1$ maturities of long-term AD assets) to imply exactly the one-period $\overline{S} \times \overline{S}$ AD price matrix $\overline{\mathbf{A}}$. The inadvertent loss of information in the latter recovery (compared to the former) gives rise to the consistency issue between their exact recovery results as elaborated in Equation (24), Section 3.1. Then a possible approach to mitigate the consistency issue would employ more price data than needed to obtain approximate (instead of exact) recovery results in different specifications

¹³Note that when this condition holds, we have $H \equiv \frac{1 - \sum_{\{j\}=\{1\}}^{\{K\}} p_{0,\tau}(1,j)}{1 - \sum_{\{j\}=\{1\}}^{\{K\}} p_{0,\tau}(1,j) \frac{M_j}{M_1}} = 1$, and systems (118) and (105) are identical in this case. Therefore, we do not need the additional condition H = 1 to identify (118) with (105).

that might exhibit no mutual inconsistencies.

In this approximate recovery approach, we transform and interpret the recovery systems (4) and (105) as regression equation systems, employing all T available (possibly redundant) price data inputs to uniquely obtain a set of best-fit characteristics (i.e., recovery results associated with the given subjective specification).¹⁴ So the best-fit recoveries can flexibly accommodate as many available maturities T as data sources provide and also addresses an important conceptual question on whether observing an infinite amount of error-free price data always assures successful recoveries. **Best-fit Ross's recovery**: For the original recovery system (4), the best-fit one-period AD asset price matrices associated with the original S and consolidated \overline{S} specifications are

Original system:
$$\mathbf{A} = [\mathbf{A}'_T \mathbf{A}_T]^{-1} \mathbf{A}'_T \mathbf{A}_{T+1},$$

Consolidated system: $\overline{\mathbf{A}} = [\overline{\mathbf{A}}'_T \overline{\mathbf{A}}_T]^{-1} \overline{\mathbf{A}}'_T \overline{\mathbf{A}}_{T+1},$ (119)

where \mathbf{A}_T and $\overline{\mathbf{A}}_T$ are matrices of containing all available long-term AD asset prices and consolidated long-term AD asset prices associated respectively with S and \overline{S} .¹⁵ Employing more price data does not weaken the consistency conditions, and hence does not alleviate the consistency issue in the basic recovery approach. Intuitively, this is because the no-arbitrage condition applies uniformly across all price data points employed in the consolidation, as reflected in the presence of a single indicator matrix \mathbf{C} in (16) across various data horizons. As the AD price matrix \mathbf{A} in the original specification S is exogenous to the consolidated specification \overline{S} (and thus \mathbf{C}), the consistency condition (19) applies for both the original (just-identified) and the best-fit implementations of the basic recovery process. As a result, (21) remains a necessary and sufficient condition for the (equally elusive) best-fit approach to Ross's recovery.

Best-fit generalized recovery: For specificity, let $\{1\}$ and $\overline{1}$ be the current state in the original

$$\underbrace{\mathbf{A}_{T+1}}_{T\times S} = \underbrace{\mathbf{A}_T}_{T\times S} \underbrace{\mathbf{A}}_{S\times S}, \qquad \underbrace{\overline{\mathbf{A}}_{T+1}}_{T\times \overline{S}} = \underbrace{\overline{\mathbf{A}}_T}_{T\times \overline{S}} \underbrace{\overline{\mathbf{A}}}_{\overline{S}\times\overline{S}}$$

¹⁴To illustrate, for the specific Example 1, the exact recovery employs only four maturities ($\tau \in \{1, 2, 3, 4\}$) of AD assets for the original specification $S = \{1, 2, 3\}$, and only three maturities ($\tau \in \{1, 2, 3\}$) for the consolidated specification $\overline{S} = \{\overline{1}, \overline{2}\}$.

¹⁵The best-fit solutions (119) arise from the regression-based formulation of the recovery systems (4) when there are more data inputs than unknowns (over-identification),

S and consolidated \overline{S} specifications. For the best-fit generalized recovery, we stack equations in (105) for T different horizons in S and \overline{S} into respective matrix forms, $\mathbf{A}_T \mathbf{f} = \boldsymbol{\delta}$ and $\overline{\mathbf{A}}_T \overline{\mathbf{f}} = \overline{\boldsymbol{\delta}}$, where the explicit expressions of these matrices and vectors are as follows,

$$\begin{bmatrix} A_{11} & \dots & A_{1S} \\ A_{2;11} & \dots & A_{2;1S} \\ \vdots & \dots & \vdots \\ A_{T;11} & \dots & A_{T;1S} \end{bmatrix} \begin{bmatrix} 1 \\ \frac{M_1}{M_2} \\ \vdots \\ \frac{M_1}{M_S} \end{bmatrix} = \begin{bmatrix} \delta \\ \delta^2 \\ \vdots \\ \delta^T \end{bmatrix}, \text{ and } \underbrace{\begin{bmatrix} \overline{A}_{\overline{1}\,\overline{1}} & \dots & \overline{A}_{\overline{1}\,\overline{S}} \\ \overline{A}_{2;\overline{1}\,\overline{1}} & \dots & \overline{A}_{2;\overline{1}\,\overline{S}} \\ \vdots & \dots & \vdots \\ \overline{A}_{T;\overline{1}\,\overline{1}} & \dots & \overline{A}_{T;\overline{1}\,\overline{S}} \end{bmatrix} \begin{bmatrix} 1 \\ \frac{\overline{M}_1}{\overline{M}_2} \\ \vdots \\ \frac{\overline{M}_1}{\overline{M}_{\overline{S}}} \end{bmatrix} = \underbrace{\begin{bmatrix} \overline{\delta} \\ \overline{\delta}^2 \\ \vdots \\ \overline{\delta}^T \\ \vdots \\ \overline{\delta}^T \end{bmatrix}}_{\overline{\delta}}.$$
(120)

We assume that the original and consolidated price matrices \mathbf{A}_T , $\overline{\mathbf{A}}_T$ have full rank given the long time horizon and randomness in the price data. Note that the generalized recovery equations (120) are non-linear (involving powers of time discount factors) and the recovered time discount factors' consistency condition requires that $\overline{\delta} = \delta$. For a consistency analysis, at first we take $\overline{\delta} = \delta$ as required (assuring the time discount factors' consistency), which allows to interpret (120) as linear equation systems that solve for ratios of marginal utilities, $\mathbf{A}_{\tau}\mathbf{f} = \delta$ and $\overline{\mathbf{A}}_{\tau}\mathbf{f} = \overline{\delta}$. When employed price data are redundant, T > S and $T > \overline{S}$, the best-fit recovery solutions of the marginal utilities are respectively,

$$\mathbf{f} = \begin{bmatrix} \mathbf{A}_{\tau}' \mathbf{A}_{\tau} \end{bmatrix}^{-1} \mathbf{A}_{\tau}' \boldsymbol{\delta}, \qquad \overline{\mathbf{f}} = \begin{bmatrix} \overline{\mathbf{A}}_{\tau}' \overline{\mathbf{A}}_{\tau} \end{bmatrix}^{-1} \overline{\mathbf{A}}_{\tau}' \overline{\boldsymbol{\delta}} = \begin{bmatrix} \overline{\mathbf{A}}_{\tau}' \overline{\mathbf{A}}_{\tau} \end{bmatrix}^{-1} \overline{\mathbf{A}}_{\tau}' \boldsymbol{\delta}.$$
(121)

Using the relationship $\overline{\mathbf{A}}_T = \mathbf{A}_T \mathbf{C}$ between long-term AD asset prices in the two specifications, we have¹⁶

$$\begin{bmatrix} \mathbf{C}' \mathbf{A}_{\tau}' \mathbf{A}_{\tau} \mathbf{C} \end{bmatrix} \bar{\mathbf{f}} = \mathbf{C}' \mathbf{A}_{\tau}' \boldsymbol{\delta}.$$
(122)

We observe that the solution $\overline{\mathbf{f}}$ to the second equation in (121) (or equivalently (122)) does not

¹⁶Indeed, $\overline{\mathbf{A}}_T = \mathbf{A}_T \mathbf{C}$ implies that the left-hand side of (122) is $[\mathbf{C}' \mathbf{A}'_{\tau} \mathbf{A}_{\tau} \mathbf{C}] \overline{\mathbf{f}} = [\overline{\mathbf{A}}'_{\tau} \overline{\mathbf{A}}_{\tau}] \overline{\mathbf{f}}$. The substitution of $\overline{\mathbf{f}}$ from (121) shows that this expression equals $[\overline{\mathbf{A}}'_{\tau} \overline{\mathbf{A}}_{\tau}] [\overline{\mathbf{A}}'_{\tau} \overline{\mathbf{A}}_{\tau}]^{-1} \overline{\mathbf{A}}'_{\tau} \delta = \overline{\mathbf{A}}'_{\tau} \delta = \mathbf{C}' \mathbf{A}'_{\tau} \delta$, which is the right-hand side of (122).

present a valid marginal utility recovery result in general,¹⁷, indicating a consistency issue. However, when condition (21) holds, the two equation systems in (120) yield consistent solutions for time discount factors and ratio of marginal utilities.

In summary, the best-fit approach that utilizes redundant (possibly, infinite) price data does not alleviate the consistency issue in either the original or generalized recovery approaches, except when a strong necessary and sufficient condition underlying Proposition 1 holds.

 $^{^{17}}$ This is because the first entry of such a solution generally differs from one, hence violating the normalization constraint of a valid recovery solution (per the second system in (120))