Recovery and Consistency*

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Abstract

Recovery, the process of uniquely determining market's belief, time and risk preferences from asset prices, requires a subjective state-space specification of the underlying economy that is not observed before the recovery is implemented. Different subjective specifications lead to different recovery results that, albeit unique under the respective specifications, are almost surely inconsistent with each other. This consistency issue prevails universally in the original, extended, and approximate versions of the recovery, given perfect (error-free and infinite) price data and when all required recovery assumptions are upheld. Consistency requirement highlights a new and general challenge for the recovery paradigm.

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1 Introduction

The question of how much the observed prices of traded financial contracts can reveal about unobserved market's expectation and preference has long fascinated the finance research areas of market efficiency, rational and behavioral asset pricing, and out-of-sample forecasting. Ross (2015)'s Recovery Theorem advances an ambitious approach to address this question by identifying sufficient conditions on cashflow and pricing dynamics, under which today's price data of assets across different maturities suffice to simultaneously and uniquely determine the market's preferences and transition probabilities across states. While the Recovery Theorem's conceptual formulation is elegant, its applicability depends foremost on the implementability of the theorem in practice and the empirical merit of its underlying sufficient conditions.

The current paper investigates various implementability aspects of the Recovery Theorem's original and generalized approaches, in both discrete and continuous state-time settings. We observe that the recovery implementation necessarily posits a specification of the underlying market's state space at the onset and then proceeds to recover the market's belief and preference contingent on this state space specification. The state space specification is not observed prior to the recovery implementation and therefore is endogenous in the recovery process. An endogenous state space specification has profound implications on recovery results. We demonstrate that different sets of results recovered under different state space specifications are almost surely inconsistent with each other even when every set of results is recovered uniquely under the respective specification. That is, the same key innovation of employing different asset maturities and their prices to determine preferences and probabilities of different states that defines the recovery process also makes the process almost surely inconsistent. This inconsistency result holds universally for the original, extended, and approximate versions of the recovery approach, when price data are perfect and all required assumptions of the Recovery Theorem are upheld.

The classic approach of Breeden and Litzenberger (1978) to recover the state transition dynamics from asset prices is associated with the risk-neutral premise, in which the discount factor is the risk-free rate. As a result, this approach recovers a risk-neutral transition probability, but not the one associated with the expectation and belief of the underlying market. When the discount factor is not presumed, even if financial markets are complete, many combinations of risk aversion and transition probabilities (i.e., belief) are consistent with asset prices, making an unambiguous (unique) recovery of the market's risk preference and belief impossible. To alleviate this ambiguity, Ross (2015)'s Recovery Theorem posits on the special class of time separable preferences and homogeneous transition probabilities and demonstrates a unique recovery under these assumptions. Guided by the Recovery Theorem, the implementation of the recovery paradigm specifies a state space and employs price data to recover the preference and transition probabilities conditional on the specified state space.

Central and novel in our analysis is the concept of consistency in the recovery paradigm. Intuitively, state space specification is a subjective input in the recovery process. Two analysts implement the recovery conditional on two different subjective state space specifications and, under the recovery assumptions, obtain two respective unique sets of recovery results. An important question is whether these results are consistent with each other and imply the same underlying market preference and belief. Our paper addresses this key question in three aspects. First, we quantify the consistency conditions for the recoveries under different subjective state space specification conditions. Second, we establish a necessary and sufficient condition for the consistency of recoveries under different specifications to hold. Third, we demonstrate the origin of recovery consistency from a continuous state-time setting perspective, in which the consistency issue arises from the mismatch between the state space specification and data sampling frequency. Simulation results illustrate and provide supporting evidences for our analysis. We briefly discuss these aspects in order next.

First, when the recovery works, the consistency of recovery results associated with different state space specifications is the general requirement that these results arise from the same underlying market. As an illustration, when several states of the first specification correspond to a single (consolidated) state of the second specification, the transition probabilities recovered under the the first specification for these states also need to correspond to the transition probability recovered under the second specification concerning the single state. Building upon this characterization, we construct general consistency conditions for different state space specifications that are subject to the same underlying market and observed asset price data.

Second, because the state space specification is not observed and hence needs to be presumed prior to the recovery, satisfying consistency conditions presents a key criterion to evaluate a recovery process. We show that the recovery consistency conditions hold if and only if every state in the first specification that corresponds to a single state in the second specification is identical in the underlying market. Furthermore, this necessary and sufficient condition for recovery consistency is universal in the sense that it applies to different (original, generalized, and approximate) versions of the recovery paradigm. This necessary and sufficient condition is intuitive. It asserts that only when states are indistinguishable by the asset market can they be aggregated and consolidated into a single state that is consistent with and hence is consistently recovered from the given price data. Evidently, this necessary and sufficient condition is strong because in general no two states are indistinguishable in the underlying market.[1] This finding identifies a restrictive nature of the recovery paradigm with respect to consistency.

Third, a continuous-time formulation of the recovery paradigm offers insights into the nature of the recovery consistency issue. In continuous settings, the underlying market and asset price dynamics follow continuous-time processes. The recovery then amounts to uniquely determining the governing parameters of these processes from price data. As an illustration, when the underlying dynamics are slow (or fast) moving, data need to be sampled at an adequate frequency to correctly reveal the market volatility. In practice, the sampling frequency in the the recovery implementation is dictated by the price data availability and hence is subjectively chosen by analysts before the underlying dynamics are recovered. As a result, when the unobserved underlying market dynamics are not compatible with the price data sampling frequencies that are subjectively chosen by different analysts, the associated recovered dynamics by analysts can be inconsistent with each other and distorted from the underlying dynamics.

The elegant formulation of the Recovery Theorem has renewed interests and scrutinies in the recovery literature. The empirical supports of the Recovery Theorem have been

[1] Because asset market is assumed to be complete in Ross's recovery, indistinguishable states are not due to incomplete markets.

mixed. Key assumptions underlying the Recovery Theorem are found to impose a strong constraint on the persistence of the underlying stochastic discount factor (SDF). This analysis is built upon the decomposition of the SDF growth into a permanent and a transitory components (Alvarez and Jermann, 2005). Conceptually, the Recovery Theorem's time-separable and time-homogeneous assumptions imply that only the transitory component of the SDF can be unambiguously determined so that the recovered SDF is close to the underlying SDF only when the latter is highly transitory. However, long-term asset (bond and equity) price data implicate a highly persistent underlying SDF, calling into question the empirical content of the Recovery Theorem's assumptions as in Borovička et al. (2016) and Hansen and Scheinkman (2016) and raising their counterfactual implications on asset prices as in Bakshi et al. (2018) and Jackwerth and Menner (2020). Qin et al. (2018) further evaluate recovery assumptions and results against respectively options on T-bond futures and bond price data, Audrino et al. (2021) investigate the predictive power of the recovered distribution, and Christensen (2017) provides an empirical framework to analyze SDF's permanent and transitory components. Complementing and supporting this literature, the consistency issue identified in the current paper presents a methodology challenge for the recovery paradigm that prevails independent of the price data quality and empirical merits of the assumptions needed for the recovery process. Altogether, these results indicate two (non mutually exclusive) possibilities. First, surveys remain the direct and informative channel to learn about investors' beliefs and their rich and complex contingent investment decisions as demonstrated by Giglio et al. (2022). Second, under some weak assumptions on market's behaviors, asset price data can only provide us with useful bounds on market's expectations as demonstrated by Martin (2017) and Gormsen and Koijen (2020).[2]

Several conceptual advances and theoretical extensions of the Recovery Theorem have been proposed. Carr and Yu (2012) demonstrate a unique recovery in a continuous state-time setting if the underlying state variables follow bounded stochastic processes. Walden (2017) generalizes the recovery to settings with an extended support of the state dynamics. Dubynskiy and Goldstein (2013) emphasize the consequential role and fragility of the

[2]Otherwise, one has to rely on adopting specific parametric models to estimate market's expectation and preferences from price data.

imposition of boundary conditions on the recovered quantities. Qin and Linetsky (2016) extend Ross's recovery to continuous-time Markov processes. Huang and Shaliastovich (2014) consider recovery in a recursive utility framework. Dillschneider and Maurer (2019) discuss the Perron-Frobenius operator theory in recovery. Martin and Ross (2019) discuss the relationship between the recovered time preference and the unconditional expected return on long-maturity bonds. Jensen, Lando and Pedersen (2019) relax the time homogeneous assumption and substantially generalize the Recovery Theorem's premise to allow for growing state spaces. Importantly, by quantifying the recovery process as a system of nonlinear equations, this generalized recovery approach enables an analytical characterization of the recovery's success (and impossibility) as an event of measure one (and zero). Building on this literature, our paper considers both discrete and continuous settings and examines the recovery consistency issue for both Ross's and generalized recovery. Recall that the recovery is consistent only for a specific set of market configurations, where the states in one state space specification corresponding to a single state in the second specification are identical. Therefore, following the generalized recovery's analytical characterization, the consistency issue arises with measure one in the recovery implementation process.

The current paper is organized as follows. Section 2 briefly describes Ross's recovery and the generalized recovery approaches in a discrete setting and discusses the consistency aspect of the recovery process. Section 3 analyzes the consistency of the Recovery Theorem. Section 4 analyzes the consistency of the generalized recovery. Section 5 analyzes the recovery consistency in continuous settings. Section 6 concludes. The attached Internet Appendix presents technical derivations omitted from the main text.

2 Recovery Theorem: Preliminaries

We first briefly describe the basic and extended setups of the recovery paradigm, before discussing various aspects of the recovery implementation.

2.1 Discrete State-Time setting

Consider a no-arbitrage pricing framework in complete markets and discrete state and time setting. The pair (i, t) specifies the state and time, with $i \in \mathcal{S} \equiv \{1, \ldots, S\}$ and $t \in \{0, \ldots, T\}$. Let $P_{it}(x)$ be the price at the state and time (i, t) of a state-dependent payoff realized at $t + 1 \leq T$ and denoted by vector $x = (x_1, \ldots, x_S)$. The no-arbitrage pricing of the payoff in the physical (i.e., data-generating) measure is,

$$P_{it}(x) = E_{it}\left[M_{t,t+1}(i,j)x_j\right], \qquad \text{or} \qquad P_{it}(x) = \sum_{j=1}^{S} p_{t,t+1}(i,j)M_{t,t+1}(i,j)x_j, \tag{1}$$

where $p_{t,t+1}(i, j)$ denotes the transition probabilities in the physical measure from state i at t to state j at $t+1$, and $M_{t,t+1}$ is the stochastic discount factor (SDF) growth, for the period from t to $t + 1$. The recovery of the physical distribution of states and risk preference from asset prices relies on the following assumptions.

Assumptions:

A1. The preference is time-separable, or the SDF growth has the following functional form: $M_{t,t+1}(i, j) = \delta \frac{M_j}{M_i}$, $\forall t \in \{0, \ldots, T-1\}$, $\forall i, j \in \mathcal{S}$, *where* δ *is a constant parameter and* M_i *is a function of state* i *(but not time).*

A2. The state transition dynamics are time-homogeneous, or the transition probabilities in the physical measure are time-independent: $p_{t,t+1}(i, j) = p_{i,j}$, $\forall t \in \{0, \ldots, T-1\}$, $\forall i, j \in \mathcal{S}$.

Ross (2015) formulates and motivates (i) the first assumption from an economic perspective of a representative agent's time-separable utility function, and (ii) the second assumption from a statistical perspective of Markovian risk dynamics. Accordingly, δ and M_i characterize respectively the time discount factor and marginal utility of the representative agent.

Ross's Recovery

We consider a specific contract in complete financial markets whose payoff in state j equals the inverse of marginal utility $\frac{1}{M_j}$, $j \in \{1, \ldots, S\}$. On one hand, given the time-separable preference (Assumption A1), the pricing equation (1) applied on this contract in the physical measure reads,[3]

$$P_{it}\left(\frac{1}{M}\right) = E_{it}\left[\delta \frac{M_j}{M_i}\frac{1}{M_j}\right] = \sum_{j=1}^{S} p_{t,t+1}(i,j)\delta \frac{M_j}{M_i}\frac{1}{M_j} = \delta \frac{1}{M_i}\sum_{j=1}^{S} p_{t,t+1}(i,j) = \delta \frac{1}{M_i} = \delta x_i, \tag{2}$$

or,

$$E_{it}\left[M_{t,t+1}(i,j)x_j\right] = \delta x_i, \qquad x_j = \frac{1}{M_j}, \;\; \forall j \in \{1,\ldots,S\}. \tag{3}$$

This equation system formalizes a direct implication of Assumption A1 on the recovery process that when preferences are time-separable, the inverse marginality utilities $\{x_i\}$ constitute an eigenvector of the SDF.

On the other hand, given the state i at time $t \in \{0,\ldots,T-1\}$, let A_{ij} be the current price of the one-period Arrow-Debreu (AD) asset that is contracted on the initial state i and offers a unit payoff if the next-period state is $j \in \{1,\ldots,S\}$, and zero payoff otherwise. The above contract has an identical payoff to a portfolio of $x_j = \frac{1}{M_j}$ units of j-th AD asset, $j \in \{1,\ldots,S\}$. Therefore, its current pricing is $P_{it}(x) = \sum_{j=1}^{S} A_{ij}x_j$. Identifying this contract's price with the one obtained in (2) implies a key recovery equation,

$$\sum_{j\in\mathcal{S}} A_{ij}x_j = \delta x_i, \text{ with } x_j = \frac{1}{M_j}, \forall i,j \in \mathcal{S}, \quad \text{or} \quad \mathbf{A}\mathbf{x} = \delta\mathbf{x}, \tag{4}$$

where $\mathbf{A}$ denotes the $S \times S$ matrix of one-period AD asset prices, and $\mathbf{x}$ is the $S \times 1$ vector of inverse marginal utilities. The time-homogeneity assumption A2 assures that both $\mathbf{A}$ and $\mathbf{x}$ are time-invariant. Conceptually, this equation indicates that the time discount factor and the inverse of marginal utilities are respectively the eigenvalue and eigenvector of the one-period AD price matrix $\mathbf{A}$. The absence of arbitrages assures that the prices of all AD assets are strictly positive. An application of Perron-Frobenius theorem implies that there exists a unique eigenvector of the AD price matrix, whose elements are strictly positive and is associated with the dominant eigenvalue.[4] That is, only one eigenvector of the AD price

[3]To obtain this equation, we substitute $M_{t,t+1}(i,j) = \delta \frac{M_j}{M_i} = \delta \frac{x_i}{x_j}$ (Assumption A1) and $x_j = \frac{1}{M_j}$ (inverse marginal utility contract), $\forall i,j \in \mathcal{S}$, into the pricing equation (1).

[4]Both eigenvectors and marginal utilities are determined only up to a multiplicative factor. Therefore, the

matrix qualifies the positivity criterion of marginal utilities $\{x_j\}$. The unambiguous recovery of the representative agent's preference then amounts to determining this unique dominant eigenvector of the AD price matrix Ross (2015). The transition probability $p_{t,t+1}(i,j)$ from the current state i to a state j next period then follows from the pricing equation (1) for AD assets in the physical measure

$$A_{ij} = E_{it}\left[M_{t,t+1}(i,s)\mathbb{1}_j(s)\right] \Longrightarrow p_{t,t+1}(i,j) = \delta^{-1} A_{ij} \frac{M_i}{M_j} = \delta^{-1} A_{ij} \frac{x_j}{x_i}, \tag{5}$$

where the indicator function $\mathbb{1}_j(s)$ denotes the payoff of AD asset A_{ij}.

Empirically, in financial markets at time t, we do not observe the price $A_{\ell j}$ of AD assets that are contracted on initial states ℓ different from the current state at t. However, Assumption A2 on the time-homogeneity of the state transition dynamics enables an inference of the entire AD price matrix needed in the recovery (4). Let $A_{\tau;ij}$ be the current price of τ-period AD asset contracted on the current state i that offers a unit payoff if the state in τ periods is j and zero payoff otherwise. Prices $\{A_{\tau+1;ij}\}$, $j \in \mathcal{S}$, of $(\tau+1)$-period AD assets contracted on the same current state i are obtained from rolling over τ-period AD assets $\{A_{\tau;ij}\}$ for one more period. Noting that the one-period AD price matrix is time-invariant, assembling these pricing equations recursively for $\tau \in \{1, \dots, T\}$ yields,

$$\underbrace{\begin{bmatrix} A_{2;i1} & \dots & A_{2;iS} \\ A_{3;i1} & \dots & A_{3;iS} \\ \vdots & \dots & \vdots \\ A_{\tau+1;i1} & \dots & A_{\tau+1;iS} \end{bmatrix}}_{\equiv \mathbf{A}_{\tau+1}} = \underbrace{\begin{bmatrix} A_{i1} & \dots & A_{iS} \\ A_{2;i1} & \dots & A_{2;iS} \\ \vdots & \dots & \vdots \\ A_{\tau;i1} & \dots & A_{\tau;iS} \end{bmatrix}}_{\equiv \mathbf{A}_{\tau}} \times \underbrace{\begin{bmatrix} A_{11} & \dots & A_{1S} \\ \vdots & \ddots & \vdots \\ A_{S1} & \dots & A_{SS} \end{bmatrix}}_{\equiv \mathbf{A}}, \tag{6}$$

where the one-period AD price A_{ij} is the abbreviated version of the full notation $A_{1;ij}$, $\forall i, j$. By definition, both matrices $\mathbf{A}_\tau$ and $\mathbf{A}_{\tau+1}$ store price data of AD assets contracted on the same current state i, and hence are observable from financial markets. When enough price data is collected, the system (6) of linear equations solves for the one-period AD price matrix

uniqueness (up to a multiplicative factor) of a positive eigenvector implicates a unique (up to a multiplicative factor) set of marginal utilities in Ross's recovery.

A,[5] which is input to equation (4) in the recovery process.

To summarize, Ross (2015)'s recovery relies on both the time-homogeneity of state transition dynamics and the time-separability preference. These two assumptions effectively result in a two-stage implementation of Ross's recovery that consists of (i) the inference of the entire AD price matrix (6), and (ii) the solution to recovery equations (4). The recovery identifies the dominant eigenvalue and eigenvector of the AD price matrix respectively with the time discount factor and inverse marginal utility vector (4). Then, the one-period transition probabilities in the physical measure are recovered using (5).

Generalized Recovery

The basic recovery approach is both ambitious and intricate. On one hand, it aims to recover the transition probability in the physical measure between every two perceived states of the economy. On the other hand, it requires the inference of the entire price matrix of AD assets, relying on the time-homogeneity of the state transitions (Assumption A2). Such an inference is challenging because AD assets with initial states different from the actual current states are not observed in financial markets, and the time-homogeneity assumption rules out interesting dynamics of the state space. Jensen, Lando and Pedersen (2019) (referred to as JLP (2019) hereafter) observe these important limitations, tackle them by relaxing Assumption A2 on the time-homogeneity and establish a generalized version of the recovery. By eliminating a strong requirement for the recovery, the generalized recovery significantly widens the set of recoverable state space dynamics while focusing on the probabilities of the transitions exclusively from the actual current state of the economy and the marginal utilities.

Because the generalized recovery works exclusively with the current state (at $t = 0$), for specificity we name the current state the first state ($i = 1$). The pricing of the τ-period AD assets $\{A_{\tau;1j}\}$, for $\{j \in \mathcal{S}\}$, implies a relation between AD prices and the corresponding

[5] In subsequent sections, we refer to, and discuss in depth, the case in which the numbers of unknown AD prices and pricing equations are equal as the just-identified recovery, and the case in which the number of unknown AD prices is less than the number of pricing equations as the best-fit recovery.

transition probabilities,[6]

$$A_{\tau;1j}\frac{1}{M_j} = \delta^\tau p_{0,\tau}(1,j)\frac{1}{M_1}, \quad \forall j \in \mathcal{S}, \tau \in \{1,\dots,T\}. \tag{7}$$

Consequently, the summation over all final states j's, together with the condition $\sum_j p_{0,\tau}(1,j) = 1$, generates the key equation system for the generalized recovery,

$$\sum_{j\in\mathcal{S}} A_{\tau;1j}\frac{M_1}{M_j} = \delta^\tau, \quad \text{or} \quad A_{\tau;11} + \sum_{j=2}^{S} A_{\tau;1j}\frac{M_1}{M_j} = \delta^\tau, \ \ \forall \tau \in \{1,\dots,T\}, \tag{8}$$

which determines the time discount factor δ and risk preferences $\left\{\frac{M_j}{M_1}\right\}$, $j \in \{1,\dots,S\}$. Generalizing (5), the transition probability from the current state to any state j at time τ then follows from (7)

$$p_{0,\tau}(1,j) = \delta^{-\tau} A_{\tau;1j}\frac{M_1}{M_j}, \quad j \in \{1,\dots,S\}, \quad \tau \in \{1,\dots,T\}. \tag{9}$$

Note that in contrast to Ross's recovery, the generalized recovery employs price data of all T available horizons but limits to the initial state being the actual current state $\{1\}$. The generalized recovery centers on the counting arguments concerning the system (8) of S unknowns $\left\{\delta, \frac{M_1}{M_2}, \dots, \frac{M_1}{M_S}\right\}$ and T nonlinear equations (one for each value of τ in $\{1,\dots,T\}$).[7] When the number of unknowns is greater than or equal to that of equations, $S \geq T$, there are multiple solutions of this nonlinear equation system in general, ruling out an unambiguous (unique) recovery of the transition probabilities, time and risk preferences. Intuitively, in this case, limited price data are consistent with multiple possible configurations of preferences and beliefs of the representative agent in the economy. When $S < T$, the system does not have a solution in general, also ruling out a successful recovery. However, JLP (2019) crucially observe that when AD price data arise from a no-arbitrage asset pricing model consistent with a time-separable preference (Assumption A1), the system (8) has a unique

[6] To arrive at this relationship, note that the pricing of τ-period AD assets is, $A_{\tau;1j} = E_0\left[\delta^\tau \frac{M_{s_\tau}}{M_1}\mathbb{1}_j(s_\tau)\right] = p_{0,\tau}(1,j)\delta^\tau\frac{M_j}{M_1}$, where the indicator function $\mathbb{1}_j(s_\tau)$ denotes the payoff of AD asset $A_{\tau;1j}$.

[7] Because SDF can only be determined up to a multiplicative constant, the recovery process concerns only the ratios of marginal utilities $\left\{\frac{M_1}{M_2}, \dots, \frac{M_1}{M_S}\right\}$. Equivalently, we can normalize the marginal utility in the first state to be one, $M_1 = 1$, without loss of generality.

solution. Intuitively, in this case, price data are redundant and also consistent because they arise from the same underlying market model. As a result, data redundancy does not lead to inconsistencies in the solution of the system (8), and the generalized recovery works.

To summarize, the generalized recovery (JLP (2019)'s Proposition 1) relaxes the time-homogeneity requirement for the state transition dynamics to focus on recovering the transitions starting exclusively from the actual current state of the economy. This relaxation effectively results in a one-stage implementation of the generalized recovery that solves a nonlinear system (8) and helps address a richer set of recoverable state dynamics than Ross's recovery approach (4). The two approaches converge when the requirement on the time-homogeneity of state transition dynamics is reinstated.

2.2 Discussion

In principle, the paradigm of recovery is to identify assumptions and sufficient conditions, under which transition probabilities in the physical measure, time and risk preferences can be uniquely recovered from asset prices observed in financial markets. In practice, the recovery is subject to further scrutinies. Empirically, the assumptions needed for the recovery place direct constraints on the recoverable preference specifications. Such constraints can have testable implications and hence can be evaluated against other asset price data independently from the recovery framework. We describe the literature's findings about the preference specification impacts on the recovery below. Conceptually, the recovery requires a state space specification $\mathcal{S}$ for the recovery equation systems (4) and (8). Such a specification is endogenous to the recovery process and can give rise to incompatible recovery results when these equation systems are implemented under different state space specifications. We briefly discuss various implications of state space specifications on the recovery next, deferring a formal and detailed analysis to subsequent sections.

Preference Specifications

The key feature of the basic recovery is that the recoverability is strongly related to an eigenvalue problem of the pricing kernel and the existence of its unique all-positive (dominant) eigenvector (4) when recovery assumptions are upheld. However, while the time-separable preference assumption implies that the inverse marginal utilities form an eigenvector of the SDF, the inverse does not necessarily hold. Consider the Alvarez and Jermann (2005)'s decomposition of the SDF growth into a permanent (M^P) and a transitory (M^T) components,

$$M_{t,t+1} = M^P_{t,t+1}\, M^T_{t,t+1}, \qquad t \in \{0, \ldots, T-1\}.$$

By definition, the permanent component of SDF is a martingale in the physical measure, so its growth satisfies $E_t\left[M^P_{t,t+1}\right] = 1$, $\forall t \in \{0, \ldots, T-1\}$. When combined with the eigenvalue problem (2) characteristic to a time-separable preference, this martingale property indicates the following decomposition,[8]

$$M^P_{t,t+1}(i,j) = \frac{M_{t,t+1}(i,j)x_j}{\delta x_i}, \qquad M^T_{t,t+1}(i,j) = \frac{\delta x_i}{x_j}, \quad \forall i,j \in \mathcal{S}. \tag{10}$$

This decomposition shows that it is the transitory component M^T of the SDF that satisfies and has a time-separable functional form of Assumption A1, even though we started out with the time-separability characteristic equation (2) presumed on the full SDF M. Intuitively, while the time-separability implies that the inverse marginal utilities form an eigenvector of the SDF, the inverse is not necessarily true. That is, the eigenvalue problem (2) may implicate only the transitory component of the SDF, and the recovery principle obtains M^T uniquely.

Borovička et al. (2016) then raise and investigate an important question on whether the SDF is sufficiently transitory, $M \approx M^T$, so that the distortion stemming from the time-separability specification is minor and the recovery process works reasonably well. Their investigation employs a property that the inverse growth of the transitory component of the SDF equals the return on the long-term bond, $\frac{M^T_t}{M^T_{t+1}} = B^{\infty}_{t+1}$.[9] Therefore, if indeed the

[8] As a check, the permanent component satisfies $E_t\left[M^P_{t,t+1}\right] = \frac{1}{\delta x_i} E_t\left[M_{t,t+1}(i,j)x_j\right] = 1$ by virtue of (2).

[9] On one hand, Alvarez and Jermann (2005) show that given the existence of $\beta \in \mathbb{R}$ such that $\lim_{k\to\infty} \frac{E_t[M_{t,t+k}]}{\beta^k} \in (0,\infty)$, the permanent component of the SDF is $M^P_t = \lim_{k\to\infty} \frac{E_t[M_{t+k}]}{\beta^{t+k}}$. On the other hand, the return on the long-term bond in one-period holding is defined as the growth in the current value of one dollar payable in long-term future, $B^{\infty}_{t+1} = \lim_{k\to\infty} \frac{E_{t+1}[M_{t+1,t+k}]}{E_t[M_{t,t+k}]} = \frac{M_t}{M_{t+1}} \lim_{k\to\infty} \frac{E_{t+1}[M_{t+k}]}{E_t[M_{t+k}]}$. Using the permanent SDF component obtained above, we have $B^{\infty}_{t+1} = \frac{M_t}{M_{t+1}} \lim_{k\to\infty} \frac{E_{t+1}\left[M_{t+k}/\beta^{t+k}\right]}{E_t\left[M_{t+k}/\beta^{t+k}\right]} =$

SDF M_t is mostly transitory, then its growth $M_{t,t+1}$ almost perfectly (negatively) correlates with the long-term bond return. As a result, among all traded assets, the long-term bond should offer the largest (absolute) Sharpe ratio. The empirical fact that the long-term bond's Sharpe ratio is not superior (in the absolute value) compared to that of other financial asset classes then rules out the dominance of the transitory component M^T in the SDF. In sum, Borovička et al. (2016)'s analysis indicates a counterfactual empirical aspect of the time-separable preference specification assumed in the recovery approach. The analysis applies to both original and generalized versions of the recovery because both versions make use of the time-separable preference assumption.

State-Space Specifications

Given the required assumptions, the recovery implementation proceeds from price data for T horizons and a specification of S states in systems (4) and (8). Price data and their extent T are observable and amenable to the data availability, whereas the specification S of the objective underlying market's state space is not observed and is up to the analyst's discretion. It is then important to understand the impacts of the state space specification on the recovery and map results recovered under different specifications. A thought experiment demonstrates such a mapping. Consider two analysts adopting two different and subjective state space specifications, $\mathcal{S} = \{1, \ldots, S\}$ of S states and $\overline{\mathcal{S}} = \{\overline{1}, \ldots, \overline{S}\}$ of $\overline{S}$ states. The comparison of recoveries results under different specifications involves several aspects, including the consolidation of the state space and the consistency of recovery results. To illustrate these aspects at first, we employ a simple example (Example 1 and Figure 1 below).

Consolidation: Without loss of generality, let $\overline{S} < S$. We refer to $\mathcal{S}$ as the original specification and $\overline{\mathcal{S}}$ as the consolidated specification denoted by an overline sign. A consolidated state $\overline{j} \in \overline{\mathcal{S}}$ is characterized either as (i) a single consolidated state if it is identical to an original state $\overline{j} \equiv j$, or (ii) a coupled consolidated state if it is composed of multiple original states $\overline{j} \supset j$. In the analysis, these specifications are assumed and adopted by different analysts attempting to recover the underlying preferences and state transition probabilities

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$\frac{M_t}{M_{t+1}} \frac{M^P_{t+1}}{M^P_t} = \frac{M^T_t}{M^T_{t+1}}$, where we have used the decomposition $M_t = M^P_t M^T_t$ in the last equality.

of the market. It is important to note that our references of the *original* and *consolidated* specifications are purely conventional and are subjective specifications adopted by different analysts.

Example 1 (State Space Illustration) *The first (original) state space specification has three states* $\mathcal{S} = \{1, 2, 3\}$*, and the second (consolidated) has two states* $\overline{\mathcal{S}} = \{\overline{1}, \overline{2}\}$*. The first consolidated state is identical to the first original state,* $\overline{1} = \{1\}$*, i.e.,* $\overline{1}$ *is a single state. The second consolidated state is composed of the two remaining original states,* $\overline{2} = \{2, 3\}$*, i.e.,* $\overline{2}$ *is a coupled state.*

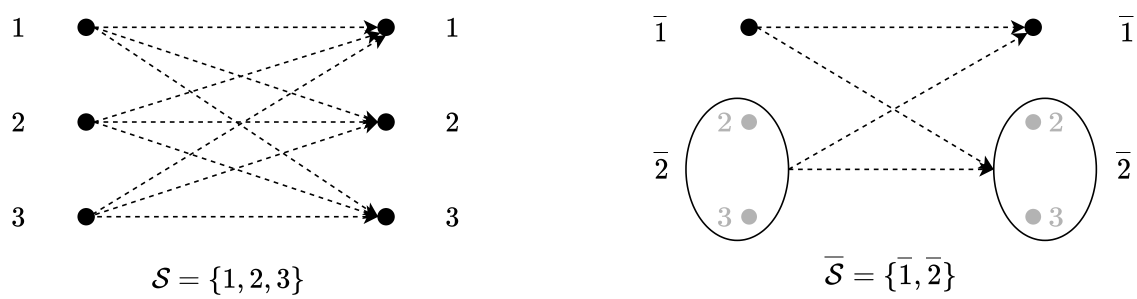


Figure 1: An illustration of different state space specifications.

To reconcile the recoveries based on different specifications, for a single state, we directly compare the recovery results under $\mathcal{S}$ and $\overline{\mathcal{S}}$. For a coupled state, a consolidation process is needed for the comparison. We first aggregate the recovery results over all states under specification $\mathcal{S}$ that correspond to the coupled state under $\overline{\mathcal{S}}$ before making the comparison. For the specific Example 1, the consolidation and comparison are as follows. We directly compare the recovery results (i.e., transition probabilities and marginal utilities) for the single state $\{1\}$ under the original specification $\mathcal{S}$ and $\overline{1}$ under the consolidated specification $\overline{\mathcal{S}}$, and aggregate and compare the recovery results for states $\{1, 2\}$ under $\mathcal{S}$ and the coupled state $\overline{2}$ under $\overline{\mathcal{S}}$.

Consistency: Recoveries under different specifications $\mathcal{S}$ and $\overline{\mathcal{S}}$ are subject to consistency criteria, given that these recoveries address the same underlying objective market's transition probabilities and risk and time preferences. Intuitively, the recovery consistency amounts to recovering identical values for (i) time discount factors, (ii) marginal utilities in single states, and (iii) probabilities for transitions among single states (and coupled states, after the appropriate consolidation is performed). The recovery consistency conditions can be obtained by comparing different state space specifications and the associated AD prices observed and perceived by respective analysts. For an illustration, consider the specific Example 1 and suppose that the current state is 1 (perceived by the first analyst), which is also $\overline{1}$ (perceived by the second analyst). The consistency conditions for the time discount factors and transition probabilities recovered by the two analysts are implied directly from their subjective state space specifications $\mathcal{S}$ and $\overline{\mathcal{S}}$ (per Figure 1)

$$\overline{\delta} = \delta, \qquad \overline{p}_{t,t+1}(\overline{1}, \overline{1}) = p_{t,t+1}(1,1), \qquad \overline{p}_{t,t+1}(\overline{1}, \overline{2}) = p_{t,t+1}(1,2) + p_{t,t+1}(1,3), \tag{11}$$

for all t. The consistency conditions for the recovered marginal utilities are implied from the pricing of traded assets. The two analysts observe AD asset prices traded in markets and interpret these contractual prices in accordance with their subjective specifications. Specifically, because current states are identical for analysts ($\overline{1} = \{1\}$), they interpret an identical AD price contracted on these states $\overline{A}_{\overline{1}\,\overline{1}} = A_{11}$. As non-current states are different for analysts ($\overline{2} = \{2, 3\}$), they interpret different AD prices contracted on these states. Instead, these prices are related as $\overline{A}_{\overline{1}\,\overline{2}} = A_{12} + A_{13}$. Substituting AD prices from the pricing equation (7) into this AD asset price relation implies the consistency condition for the recovered marginal utilities,

$$\frac{\overline{M}_{\overline{2}}}{\overline{M}_{\overline{1}}} = \frac{\frac{M_2}{M_1} p_{t,t+1}(1,2) + \frac{M_3}{M_1} p_{t,t+1}(1,3)}{p_{t,t+1}(1,2) + p_{t,t+1}(1,3)}. \tag{12}$$

We refer to Internet Appendix A.1 and A.2 for a formal derivation of the general consistency conditions for Ross's and generalized recovery approaches.

When the consistency conditions are violated, the two sets of recovered quantities are incompatible, so that at least one of them is also incompatible with the set of objective underlying parameters of the market. Since these objective underlying parameters are not observed, without additional falsification criteria it is impossible to rule out (or rule in) either specifications employed by the two analysts. The inconsistency between the two recovery specifications may also give rise to arbitrage opportunities between recovered quantities, the violation of probability law, and negative time discount factors. Therefore, the consistency conditions are crucial in relating recovery results under different state space specifications. They present a criterion to evaluate a recovery approach.

Best-fit recoveries: State space specifications are essential to the recovery because they determine the number of unknown parameters to be recovered. Different specifications concern different sets of unknowns, so a comparison between specifications involves a change in the numbers of unknowns. Given the same price data availability and constraints that all analysts face, it is impossible almost surely that two analysts with different perceived specifications are subject to the same balance of data-driven constraints versus unknowns in the recovery systems (4) and (8). It is the exact nature (i.e., counting arguments) of these systems that gives rise to both an unambiguous (unique) recovery when the above balance prevails and specification-dependent (incompatible) recoveries when this balance does not hold.

The alleviation of strict counting-argument constraints on the exact recoveries requires a flexible accommodation of different amounts of data inputs and unknown parameters. Along this approach, a best-fit recovery transforms and interprets the recovery systems (4) and (8) as regression equation systems (Sections 3.3 and 4.2). For a specification, these regression systems employ a flexible number of constraints (data inputs) to uniquely determine a best-fit set of unknown parameters of the specification (recovery results). For the specific Example 1, the just-identified recovery takes only three data horizons ($T \in \{1, 2, 3\}$) under the original specification $\mathcal{S} = \{1, 2, 3\}$ and only two ($T \in \{1, 2\}$) under the consolidated $\overline{\mathcal{S}} = \{\overline{1}, \overline{2}\}$ (see (6)). Best-fit recoveries flexibly allow as many horizons as data sources provide, which also allows us to address an important conceptual question on whether observing an infinite amount of error-free price data always assures successful recoveries.

3 Ross's Recovery

This section presents a formal analysis and simulation results of various identification aspects of the original Ross's recovery. Our analysis focuses on the comparison and consistency of the recovery results under different state space specifications of the same underlying risk pricing model. Section 3.1 introduces the comparative analysis setup, which are employed to analyze Ross's original recovery approach in Section 3.2, its best-fit version in Section 3.3, and simulation results in Section 3.4.

3.1 Comparative analysis setup

The comparative analysis concerns two different analysts. The first analyst perceives an original state space specification $\mathcal{S} \equiv \{1, \ldots, S\}$ of S states. The second analyst perceives a consolidated state space specification $\overline{\mathcal{S}} \equiv \{\overline{1}, \ldots, \overline{S}\}$ of $\overline{S}$ states. For the sake of clarity, we assume that the consolidated state space specification is netted in the original specification in the sense that every consolidated state $\overline{j}$ is composed of $S_{\overline{j}}$ original states with $S_{\overline{j}} \geq 1$,

$$S = \sum_{\overline{j}}^{\overline{S}} S_{\overline{j}}.$$

As a result, for the netted partitions, the consolidated specification is associated with unambiguously less (coarser) information about the structure of the state space than the original specification in our analysis,[10]

$$\overline{\mathcal{S}} \subset \mathcal{S} \qquad \text{and} \qquad \overline{S} < S. \tag{13}$$

The analysts implement their recovery procedures given their respective state space specifications $\mathcal{S}$ and $\overline{\mathcal{S}}$.

Given any consolidated state $\overline{j} \in \overline{\mathcal{S}}$, an original state k either is a component state of $\overline{j}$ ($k \subset \overline{j}$) or does not belong to $\overline{j}$ ($k \not\subset \overline{j}$). This binary relationship is quantified by an

[10] Our main findings on the comparability of recovery results under different state space specifications hold when these specifications are non-netted.

indicator coefficient $\mathbf{C}_{k\bar{j}}$ defined as follows,

$$\begin{cases} \mathbf{C}_{k\bar{j}} = 1 & \text{if } k \in \mathcal{S} \text{ is a component state of } \bar{j}, \\ \mathbf{C}_{k\bar{j}} = 0 & \text{if } k \in \mathcal{S} \text{ is not a component state of } \bar{j}, \end{cases} \qquad \forall \bar{j} \in \overline{\mathcal{S}}. \tag{14}$$

These indicator coefficients together form an $S \times \overline{S}$ indicator matrix $\mathbf{C}$ that fully characterizes the mapping from the original specification $\mathcal{S}$ to the consolidated specification $\overline{\mathcal{S}}$. For the specific Example 1, the indicator matrix that relates $\mathcal{S} = \{1, 2, 3\}$ and $\overline{\mathcal{S}} = \{\bar{1}, \bar{2}\}$ is

$$\left.\begin{aligned} \bar{1} &= \{1\} \\ \bar{2} &= \{2, 3\} \end{aligned}\right\} \implies \mathbf{C} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}. \tag{15}$$

3.2 Ross's Recovery: Just-Identified Approach

As described in Section 2.1, the original implementation of Ross (2015)'s recovery consists of two stages, i.e., inferring the AD matrix from price data and then solving for the dominant eigenvector and eigenvalue of that matrix. This just-identified recovery approach employs just enough AD price data to infer the entire AD matrix in the first stage (equation (6) and Section 2.2).

First stage: For the original specification of S states, observed price data for $S+1$ horizons $\tau \in \{1, \ldots, S+1\}$ contained in price matrices $\mathbf{A}_\tau$ and $\mathbf{A}_{\tau+1}$ (6) suffice to solve for the $S \times S$ AD matrix $\mathbf{A}$. For the consolidated specification of $\overline{S}$ states, observed price data for $\overline{S}+1$ horizons suffice to solve for the $\overline{S} \times \overline{S}$ consolidated AD matrix $\overline{\mathbf{A}}$. That is,

$$\text{Original system: } \underbrace{\mathbf{A}_{\tau+1}}_{S\times S} = \underbrace{\mathbf{A}_\tau}_{S\times S}\underbrace{\mathbf{A}}_{S\times S}; \qquad \text{Consolidated system: } \underbrace{\overline{\mathbf{A}}_{\tau+1}}_{\overline{S}\times\overline{S}} = \underbrace{\overline{\mathbf{A}}_\tau}_{\overline{S}\times\overline{S}}\underbrace{\overline{\mathbf{A}}}_{\overline{S}\times\overline{S}}. \tag{16}$$

The original specification requires $S+1$ horizons ($\tau \in \{1, \ldots, S+1\}$), while the consolidated specification requires $\overline{S}+1$ horizons ($\tau \in \{1, \ldots, \overline{S}+1\}$), of price data in the just-identified recovery approach. As the consolidated specification is associated with a coarser partition of the state space compared to the original specification, it requires less data for the recovery.

To relate the recovery processes under the two specifications, consider a consolidated state $\overline{j} \in \overline{\mathcal{S}}$ that is composed of several original states $j \in \mathcal{S}$. In the absence of arbitrage opportunities in asset markets, given the same initial current state i, the price of an AD asset paying off in the consolidated state $\overline{j}$ (perceived by the second analyst) must equal the sum of prices of AD assets paying off in the associated original states j (perceived by the first analyst),

$$\overline{A}_{\tau;i\overline{j}} = \sum_{j \subset \overline{j}} A_{\tau;ij}, \qquad \forall \tau \in \{1, \dots, T\}. \tag{17}$$

Assembling these no-arbitrage relations for all consolidated states $\overline{j}$ connects the observed price matrices (6) associated with the original and consolidated specifications as follows,

$$\underbrace{\overline{\mathbf{A}}_{\tau}}_{\overline{S}\times\overline{S}} = \underbrace{\begin{bmatrix} \mathbb{I}_{\overline{S}\times\overline{S}} & \mathbb{O}_{\overline{S}\times(S-\overline{S})} \end{bmatrix}}_{\overline{S}\times S} \underbrace{\mathbf{A}_{\tau}}_{S\times S} \underbrace{\mathbf{C}}_{S\times\overline{S}}; \qquad \underbrace{\overline{\mathbf{A}}_{\tau+1}}_{\overline{S}\times\overline{S}} = \underbrace{\begin{bmatrix} \mathbb{I}_{\overline{S}\times\overline{S}} & \mathbb{O}_{\overline{S}\times(S-\overline{S})} \end{bmatrix}}_{\overline{S}\times S} \underbrace{\mathbf{A}_{\tau+1}}_{S\times S} \underbrace{\mathbf{C}}_{S\times\overline{S}}, \tag{18}$$

where $\overline{S} \times S$ matrix $\begin{bmatrix} \mathbb{I}_{\overline{S}\times\overline{S}} & \mathbb{O}_{\overline{S}\times(S-\overline{S})} \end{bmatrix}$ is composed of two blocks[11] and acts to drop the extra horizons $\tau \in \{\overline{S}+2, \dots, S+1\}$ of price data not needed in the just-identified recovery under in the consolidated specification (see (16)). The $S \times \overline{S}$ indicator matrix $\mathbf{C}$, defined in (14), maps out the consolidation between two specifications $\mathcal{S}$ and $\overline{\mathcal{S}}$.

Second stage: Ross (2015)'s recovery solves and identifies the dominant eigenvector of the AD matrix with the (inverse of) marginal utilities in the second stage (4). Specifically, the recoveries under original and consolidated specifications concern the dominant eigenvectors of matrices $\mathbf{A}$ and $\overline{\mathbf{A}}$ in (16). Therefore, relating these dominant eigenvectors is important to compare and reconcile recovery results under the respective specifications $\mathcal{S}$ and $\overline{\mathcal{S}}$. To ease the exposition, we next present basic arguments to achieve a key condition to relate and reconcile these dominant eigenvectors. We relegate the technical derivation of the necessary and sufficient nature of this to Internet Appendix B.1.

Relating Ross's recovery results under different specifications starts with multiplying the

[11] As the notation indicates, the first block $\mathbb{I}_{\overline{S}\times\overline{S}}$ is an identity matrix, the second block $\mathbb{O}_{\overline{S}\times(S-\overline{S})}$ is a matrix of all zero entries.

matrix $\overline{S} \times S$ matrix $\left[\mathbb{I}_{\overline{S}\times\overline{S}} \quad \mathbb{O}_{\overline{S}\times(S-\overline{S})}\right]$ to the left, and the indicator matrix $\mathbf{C}$ to the right, of the original system in (16),

$$\left[\mathbb{I}_{\overline{S}\times\overline{S}} \quad \mathbb{O}_{\overline{S}\times(S-\overline{S})}\right] \mathbf{A}_{\tau+1}\mathbf{C} = \left[\mathbb{I}_{\overline{S}\times\overline{S}} \quad \mathbb{O}_{\overline{S}\times(S-\overline{S})}\right] \mathbf{A}_{\tau}\mathbf{AC}. \tag{19}$$

Using the consolidation (18), above equation becomes

$$\overline{\mathbf{A}}_{\tau+1} = \left[\mathbb{I}_{\overline{S}\times\overline{S}} \quad \mathbb{O}_{\overline{S}\times(S-\overline{S})}\right] \mathbf{A}_{\tau}\mathbf{AC}. \tag{20}$$

Let us denote a $\overline{S} \times \overline{S}$ matrix $\mathbf{B}$ that satisfies $\mathbf{AC} = \mathbf{CB}$. As a result, above equation is further simplified to

$$\overline{\mathbf{A}}_{\tau+1} = \left[\mathbb{I}_{\overline{S}\times\overline{S}} \quad \mathbb{O}_{\overline{S}\times(S-\overline{S})}\right] \mathbf{A}_{\tau}\mathbf{CB} = \overline{\mathbf{A}}_{\tau}\mathbf{B}, \tag{21}$$

where the second equality arises from (18). Comparing (21) with the the consolidated system in (16) indicates that the matrix $\mathbf{B}$ introduced above is identical to the AD price matrix $\overline{\mathbf{A}}$ in the consolidated identification. As a result, the defining identity of $\mathbf{B}$ (introduced below (20)) must also be satisfied by $\overline{\mathbf{A}}$, or

$$\underbrace{\mathbf{A}}_{S\times S}\underbrace{\mathbf{C}}_{S\times\overline{S}} = \underbrace{\mathbf{C}}_{S\times\overline{S}}\underbrace{\overline{\mathbf{A}}}_{\overline{S}\times\overline{S}}. \tag{22}$$

Recall that AD matrices $\mathbf{A}$ and $\overline{\mathbf{A}}$ (16) are inferred from objective price data, whereas the indicator matrix $\mathbf{C}$ (14) arises exclusively from analysts' subjective specifications. Given price data, not every indicator matrix $\mathbf{C}$ satisfies (22). Equivalently, we observe that not every two exogenous specifications of the state space are simultaneously consistent with the observed price data. Equation (22) presents an important condition for two state space specifications to be consistent in the recovery process. The specific Example 1 illustrates this observation. The substitution of indicator matrix $\mathbf{C}$ from (15) into (22) generates the following consistency conditions

$$\begin{cases} A_{11} = \overline{A}_{\overline{1}\,\overline{1}}; & A_{12} + A_{13} = \overline{A}_{\overline{1}\,\overline{2}} \\ A_{21} = \overline{A}_{\overline{2}\,\overline{1}}; & A_{22} + A_{23} = \overline{A}_{\overline{2}\,\overline{2}} \\ A_{31} = \overline{A}_{\overline{2}\,\overline{1}}; & A_{32} + A_{33} = \overline{A}_{\overline{2}\,\overline{2}}. \end{cases} \tag{23}$$

The first two conditions above relate the AD prices observed and interpreted by the two analysts in Figure 1's consolidation scheme $\overline{1} = \{1\}$ and $\overline{2} = \{2, 3\}$, whereas other conditions have more subtle implications. They require that $A_{21} = A_{31}$ as both of which are identified with $\overline{A}_{\overline{2}\,\overline{1}}$, and $A_{22} + A_{23} = A_{32} + A_{33}$ as both of which are identified with $\overline{A}_{\overline{2}\,\overline{2}}$. These conditions are both exogenous and strong to the price data. They are exogenous because they arise from subjective state space specifications in the recovery analysis, but not data per se. They are stringent because they have strong implications on the underlying asset pricing model. In fact, employing the pricing equation (7), the relations of AD asset prices (23) hold only when the original states $\{2\}$ and $\{3\}$ that belong to the same consolidated state $\overline{2}$ are identical in the underlying market model (see also Internet Appendix B.1),

$$M_2 = M_3. \tag{24}$$

Finally, from the other direction, assuming that the consistency condition (22) (or (24)) holds, we can explicitly verify the consistency of Ross's recovery results.[12] Let $S \times 1$ vector $\mathbf{x}$ and $\overline{S} \times 1$ vector $\overline{\mathbf{x}}$ denote these eigenvectors. Multiplying to the right of the consistency condition (22) by $\overline{\mathbf{x}}$ and then making use of the eigenequation in the consolidated specification, $\overline{\mathbf{A}}\overline{\mathbf{x}} = \overline{\delta}\overline{\mathbf{x}}$, we have

$$\mathbf{AC}\overline{\mathbf{x}} = \mathbf{C}\overline{\mathbf{A}}\overline{\mathbf{x}} \quad \Longrightarrow \quad \mathbf{AC}\overline{\mathbf{x}} = \overline{\delta}\mathbf{C}\overline{\mathbf{x}}. \tag{25}$$

Comparing the last equation with the eigenequation in the original specification, $\mathbf{Ax} = \delta\mathbf{x}$, indicates the following relationships between the dominant eigenquantities,

$$\mathbf{x} = \mathbf{C}\overline{\mathbf{x}}, \qquad \delta = \overline{\delta}. \tag{26}$$

[12] Recall that these results are reflected in the dominant eigenvectors and eigenvalues of the AD matrices $\mathbf{A}$ and $\overline{\mathbf{A}}$.

The relationships confirm the consistency between Ross's recovery results under two different specifications when the condition (22) holds. As the dominant eigenvector represents the inverse of marginal utilities, the first equation in (26) implies an equality between marginal utilities in a consolidated state and all corresponding original states: $x_{\overline{j}} = x_j, \forall \overline{j} \in \overline{\mathcal{S}}, \forall j \in \overline{j}$. The second equation implies the same time discount factor recovered under the original and consolidated specification.

We formalize the analysis above into a necessary and sufficient condition for the recovery implementation consistency and relegate a detailed derivation to Internet Appendix B.1.

Proposition 1 *Let $\mathcal{S}$ and $\overline{\mathcal{S}}$ denote two state space specifications of the same objective but unobserved market. The Ross's recovery results obtained under the two specifications are consistent if and only if the marginal utilities of all single states $\{j\}$ under $\mathcal{S}$ that correspond to a coupled state $\overline{j}$ under $\overline{\mathcal{S}}$ are equal.*

$$\textit{Recoveries results under } \mathcal{S} \textit{ and } \overline{\mathcal{S}} \textit{ are consistent} \Longleftrightarrow M_i = M_k,\ \forall i, k \in \overline{j},\ \overline{j} \in \overline{\mathcal{S}}. \tag{27}$$

This necessary and sufficient condition holds regardless of whether the current state is a single or coupled state under the consolidated specification.

When the necessary and sufficient condition holds, the consistent recovery results can be verified by rewriting relationships (26) explicitly using $\mathbf{C}$ (14),

$$\overline{\delta} = \delta, \qquad M_j = \overline{M}_{\overline{j}}, \quad \forall j \in \overline{j},\ \overline{j} \in \overline{\mathcal{S}}. \tag{28}$$

That is, the underlying market model needs to have identical marginal utilities in all original states $j \in \mathcal{S}$ that correspond to a same consolidated state $\overline{j} \in \overline{\mathcal{S}}$. Qualitatively, Proposition 1 highlights the endogeneity issue inherent to the recovery. The state space specification is not observed in the recovery process yet is consequential to the recovery results. Different presumed specifications lead to possibly irreconcilable recovery results. Quantitatively, Proposition 1's necessary and sufficient condition is restrictive and can only be satisfied for a special set of underlying market models. As a result, the recovery process is elusive and most likely produces inconsistent results for different analysts because their presumed, subjective specifications most likely do not satisfy condition (27). Next, we examine whether relaxing the exact Ross's recovery procedure to allow for a more robust price data input can restore the consistency in the recovery.

3.3 Ross's Recovery: Best-Fit Approach

As briefly described in Section 2.2, the best-fit recovery employs redundant (possibly all) price data to infer the entire AD matrix in the first stage (6) of Ross's recovery.

First stage: For the original specification of S states, observed price data for all T available horizons $\tau \in \{1, \dots, T\}$ contained in price matrices $\mathbf{A}_{T-1}$ and $\mathbf{A}_T$ (6) is employed to solve for the $S \times S$ AD matrix $\mathbf{A}$. For the consolidated specification of $\overline{S}$ states, observed price data for all T available horizons is also employed to solve for the $\overline{S} \times \overline{S}$ consolidated AD matrix $\overline{\mathbf{A}}$. Thus, the first-stage equation system that infers the AD matrix (6) respectively for the two specifications becomes,

$$\underbrace{\mathbf{A}_T}_{(T-1)\times S} = \underbrace{\mathbf{A}_{T-1}}_{(T-1)\times S} \underbrace{\mathbf{A}}_{S\times S}, \qquad \underbrace{\overline{\mathbf{A}}_T}_{(T-1)\times \overline{S}} = \underbrace{\overline{\mathbf{A}}_{T-1}}_{(T-1)\times \overline{S}} \underbrace{\overline{\mathbf{A}}}_{\overline{S}\times \overline{S}}. \tag{29}$$

Per the notation of (6), $\mathbf{A}_{T-1}$ and $\overline{\mathbf{A}}_{T-1}$ contain observed prices of assets having maturities $\tau \in \{1, \dots, T-1\}$, whereas $\mathbf{A}_T$ and $\overline{\mathbf{A}}_T$ having maturities $\tau \in \{2, \dots, T\}$.

Because there are presumably more constraint equations from redundant price data than the number of unknowns (entries of AD price matrices) in the above systems, the least-square (best-fit) approach is employed to infer the AD matrices,

$$\begin{aligned} \text{Original system:} \quad \mathbf{A} &= \left[\mathbf{A}'_{T-1}\mathbf{A}_{T-1}\right]^{-1} \mathbf{A}'_{T-1}\mathbf{A}_T, \\ \text{Consolidated system:} \quad \overline{\mathbf{A}} &= \left[\overline{\mathbf{A}}'_{T-1}\overline{\mathbf{A}}_{T-1}\right]^{-1} \overline{\mathbf{A}}'_{T-1}\overline{\mathbf{A}}_T. \end{aligned} \tag{30}$$

With all price data being employed, the best-fit approach addresses a practical but important issue encountered in the recovery process. That is, which data should be dropped from the recovery implementation without biases, when in principle the given price data is redundant for the determination of the AD price matrix associated with a state space specification.[13]

To relate the recovery processes under the two specifications, we observe that in the absence of arbitrage opportunities (17), the observed price matrices associated with the original and consolidated specifications satisfy

$$\underbrace{\overline{\mathbf{A}}_{T-1}}_{(T-1)\times\overline{S}} = \underbrace{\mathbf{A}_{T-1}}_{(T-1)\times S}\underbrace{\mathbf{C}}_{S\times\overline{S}}; \qquad \underbrace{\overline{\mathbf{A}}_{T}}_{(T-1)\times\overline{S}} = \underbrace{\mathbf{A}_{T}}_{(T-1)\times S}\underbrace{\mathbf{C}}_{S\times\overline{S}}, \tag{31}$$

where the $S \times \overline{S}$ indicator matrix $\mathbf{C}$ (14) characterizes the mapping between the two specifications. These equations simplify the just-identified equations (18) by not dropping any price data.

Second stage: We now present basic arguments to relate the dominant eigenvectors of AD price matrices under different state space specifications, which quantify the recovery results. By substituting the no-arbitrage relations (31) into the best-fit solution of the consolidated AD price matrix (30), we can express $\overline{\mathbf{A}}$ in terms of only original price data in $\mathbf{A}_{T-1}$ and $\mathbf{A}_T$,

$$\overline{\mathbf{A}} = \left[\mathbf{C}'\mathbf{A}'_{T-1}\mathbf{A}_{T-1}\mathbf{C}\right]^{-1}\mathbf{C}'\mathbf{A}'_{T-1}\mathbf{A}_T\mathbf{C}. \tag{32}$$

By employing the recursive relation between prices at different maturities (29), the above consolidated AD price matrix becomes

$$\overline{\mathbf{A}} = \left[\mathbf{C}'\mathbf{A}'_{T-1}\mathbf{A}_{T-1}\mathbf{C}\right]^{-1}\mathbf{C}'\mathbf{A}'_{T-1}\mathbf{A}_{T-1}\mathbf{A}\mathbf{C}. \tag{33}$$

As before, let us denote a $\overline{S} \times \overline{S}$ matrix $\mathbf{B}$ that satisfies $\mathbf{AC} = \mathbf{CB}$. Thus, the expression for the consolidated AD price matrix is simplified to

$$\overline{\mathbf{A}} = \left[\mathbf{C}'\mathbf{A}'_{T-1}\mathbf{A}_{T-1}\mathbf{C}\right]^{-1}\left[\mathbf{C}'\mathbf{A}'_{T-1}\mathbf{A}_{T-1}\mathbf{C}\right]\mathbf{B} = \mathbf{B}. \tag{34}$$

As a result, the defining identity of $\mathbf{B}$ introduced below (20) must also be satisfied by $\overline{\mathbf{A}}$, or

[13] Recall that in the original (just-identified) recovery process, when the state space is specified with S states, one needs only $S + 1$ data horizons to infer the AD price matrix in principle.

$\mathbf{AC} = \mathbf{C}\overline{\mathbf{A}}$, which is identical to the condition (22) for the just-identified implementation of the recovery. Assuming that this condition holds, the dominant eigenvectors and eigenvalues of the AD matrices $\mathbf{A}$ and $\overline{\mathbf{A}}$ satisfy (26), which then implies the consistency of recovery results under two different specifications in the best-fit implementation approach.

Employing more price data does not help to loosen the restrictive and strong effects of consistency conditions on the recovery results under different state space specifications. Intuitively, this is because the no-arbitrage condition applies uniformly across all price data points employed in the consolidation, as reflected in the presence of a single indicator matrix $\mathbf{C}$ in (18) and (31) across various data horizons. Practically, because the AD price matrix $\mathbf{A}$ under the first (original) specification $\mathcal{S}$ is largely exogenous to the second (consolidated) specification $\overline{\mathcal{S}}$ and the consolidation structure $\mathbf{C}$, the condition $\mathbf{AC} = \mathbf{C}\overline{\mathbf{A}}$ places equally non-trivial consistency constraints on the best-fit as on the just-identified implementation of the recovery process. In summary, the necessary and sufficient condition (22) and Proposition 1 also apply to the best-fit approach to Ross's recovery, which delivers consistent results under different state space specifications if and only if the marginal utilities of all single states of the original specification $\mathcal{S}$ that correspond to a coupled state of the consolidated specification $\overline{\mathcal{S}}$ are equal.

3.4 Ross's Recovery: Simulation Results

This section presents simulation evidences to illustrate Ross's recovery process under different state space specifications (Proposition 1). For the illustration of the state space specification impacts on the recovery results, simulation is a valuable approach. By design, the simulation generates perfect price data (i.e., eliminating the effect of data measurement errors on the recovery process) and assures a single underlying market model for all subjective state space specifications. Our simulation closely follows the thought experiment setup of Section 2.2 to analyze and consolidate Ross's recovery results under different specifications perceived and employed by different analysts. The simulation steps are as follows.

Simulation Procedure: Our simulation consists of $N = 20$ independent random trials. For each trial, data inputs are constructed as follows. We randomly generate (i) the time discount factor δ and marginal utility ratios $\frac{M_j}{M_1}$, $\forall j$, from continuous uniform distributions respectively with supports $[0.8, 1]$ and $\left[\frac{1}{2}, 2\right]$, and (ii) the transition probabilities as entries of a 5×5 random positive matrix with every row sum constrained to be one. For each trial, we perform the following five steps and report the corresponding simulation results.

Step 1: We take as given at first the characteristics of the underlying market model: (i) the state price specification $\mathcal{S} = \{1, \ldots, S\}$, and (ii) the time discount factor δ, the marginal utility ratios $\left\{\frac{M_j}{M_1}\right\}$ in every state $j \in \mathcal{S}$, and the one-period transition probabilities $\{p(i, j)\}$ between any two states $i, j \in \mathcal{S}$. All items in (ii) are generated in a random trial as described above.

Step 2: Using the above knowledge of objective market model and the τ-period AD pricing equation (9), we generate the current τ-period AD asset prices $\{A_{\tau;1j}\}$ that pay off in all future states $j \in \mathcal{S}$ at maturities $\tau \in \{1, \ldots, T\}$.[14] From now on we disregard the knowledge of the time and risk preferences and transition probabilities $\left\{\delta, \frac{M_j}{M_1}, p(i, j)\right\}$ of the objective underlying market model and employ only the prices $\{A_{\tau,1j}\}$ generated in this step.

Step 3: Employing the original specification $\mathcal{S}$ and AD asset prices $\{A_{\tau,1j}\}$, for $j \in \mathcal{S}$ and $\tau \in \{1, \ldots, T\}$, the first analyst recovers the time discount factor, marginal utilities and the one-period transition probabilities associated with the specification $\mathcal{S}$ in Ross's recovery process.[15] Ross's recovery process is implemented by solving (6) for the $S \times S$ one-period AD matrix $\mathbf{A}$ and diagonalizing $\mathbf{A}$ to obtain the marginal utilities and time discount factor from (4) and transition probability from (5). Without loss of generality, let $\{1\}$ denote the current state from the first analyst's perspective.

Step 4: Employing the consolidated specification $\overline{\mathcal{S}}$ and the consolidation process (17) (or

[14]Specifically, equation (9) gives the τ-period AD asset prices, $A_{\tau;1j} = \delta^\tau p_{0,\tau}(1, j)\frac{M_j}{M_1}$, in terms of the time discount factor δ, the marginal utilities $\left\{\frac{M_j}{M_1}\right\}$ and the τ-period transition probabilities $p_{0,\tau}(1, j)$. Note that the latter are computed by rolling over the one-period transition probabilities (given as the inputs in step 1) τ times

$$p_{0,\tau}(1, j) = \sum_{k_1}^{S} \sum_{k_2}^{S} \ldots \sum_{k_{\tau-1}}^{S} p(1, k_1)p(k_1, k_2) \ldots p(k_{\tau-1}, j).$$

[15]We are assuming that the original state space specification $\mathcal{S}$ perceived by the first analyst is also the specification of the underlying market model. This assumption is purely for convenience because the illustration of the recovery consistency just requires that the specifications perceived by the two analysts be different, $\mathcal{S} \neq \overline{\mathcal{S}}$.

(31)), we construct the consolidated AD asset prices $\{\overline{A}_{\tau,\overline{1}\,\overline{j}}\}$, for $\overline{j} \in \overline{\mathcal{S}}$ and $\tau \in \{1, \ldots, T\}$. Observing these prices, the second analyst then recovers the time discount factor, marginal utilities and the one-period transition probabilities associated with the specification $\overline{\mathcal{S}}$ in Ross's recovery process. That is, we repeat Step 3 while replacing the original AD asset prices $\{A_{\tau,1j}\}$ by the consolidated AD asset prices $\{\overline{A}_{\tau,\overline{1}\,\overline{j}}\}$, where without loss of generality $\overline{1}$ denotes the current state from the second analyst's perspective. For completeness, we consider two possible (and exhaustive) scenarios concerning the current state $\overline{1}$ of the consolidated specification.

Case 1: Current state $\overline{1}$ is a single state, i.e., it is identical to the current original state, $\overline{1} = \{1\} \in \mathcal{S}$.

Case 2: Current state $\overline{1}$ is a coupled state, i.e., it corresponds to several original states, $\overline{1} = \{1, \ldots, j\} \in \mathcal{S}$.

Step 5: We compare the recovery results obtained in Step 3 for the original specification $\mathcal{S}$ and Step 4 for the consolidated specification $\overline{\mathcal{S}}$ and check their consistency, i.e., verifying the holding of the consistency condition (22).

Simulation Model: Our specific simulation adopts a specification of $S = 5$ states $\mathcal{S} = \{1, 2, 3, 4, 5\}$ for the underlying market model and randomly generates and takes as given at first the time discount factor, marginal utilities, and transition probabilities (Step 1) to generate AD asset prices $\{A_{\tau;1j}\}$ (Step 2). The first analyst perceives the specification $\mathcal{S}$, uses these AD prices $\{A_{\tau;1j}\}$ for $T = 5$ horizons $\tau \in \{1, 2, 3, 4, 5\}$ to solve for the complete 5×5 one-period AD matrix $\mathbf{A}$ (6), and diagonalizes $\mathbf{A}$ to recover the time discount factor and marginal utilities (4) and transition probabilities (5) (Step 3). For every simulation trial, these recovery results are identical to the time discount factor, marginal utilities, and transition probability inputs of Step 1, validating Ross's recovery approach (4) and (5).

The second analyst perceives the a 3-state specification $\overline{\mathcal{S}} = \{1, 2, 3\}$. We consider two scenarios concerning the current state $\overline{1}$ of the consolidated specification.

$$
\begin{aligned}
&\textit{Case 1} \text{ – single current state:} \qquad && \{\overline{1}\} = \{1\},\ \overline{2} = \{2, 3\},\ \overline{3} = \{4, 5\}, \\
&\textit{Case 2} \text{ – coupled current state:} \qquad && \{\overline{1}\} = \{1, 2\},\ \overline{2} = \{3, 4\},\ \overline{3} = \{5\}.
\end{aligned}
\tag{35}
$$

For each case, (i) using (31), we construct the consolidated AD asset prices $\{\overline{A}_{\tau,\overline{1}\,\overline{j}}\}$, for $\overline{j} \in \overline{\mathcal{S}}$ and $\tau \in \{1, \ldots, T\}$, and (ii) using these AD prices $\{\overline{A}_{\tau,\overline{1}\,\overline{j}}\}$ for $T = 3$ horizons $\tau \in \{1, 2, 3\}$, the second analyst solves for the complete 3×3 one-period AD matrix $\overline{\mathbf{A}}$ (6) and then diagonalizes $\overline{\mathbf{A}}$ to recover the time discount factor and marginal utilities (4) and transition probabilities (5) associated with $\overline{\mathcal{S}}$ (Step 4).

For each case, we examine the consistency of recoveries by the two analysts by checking whether their recovery results obtained in Steps 3 and 4 satisfy the respective consistency conditions for that case (Step 5). Given the original and consolidated specifications (35), these consistency conditions read (similar to (11) and (12); see also Internet Appendix A.1),

$$\textit{Case 1} \text{ – single current state: } \begin{cases} \overline{\delta} = \delta, \quad \overline{M}_{\overline{1}} = M_1, \\ \overline{p}_{t,t+1}(\overline{1}, \overline{1}) = p_{t,t+1}(1, 1), \\ \overline{p}_{t,t+1}(\overline{1}, \overline{2}) = p_{t,t+1}(1, 2) + p_{t,t+1}(1, 3), \\ \overline{p}_{t,t+1}(\overline{1}, \overline{3}) = p_{t,t+1}(1, 4) + p_{t,t+1}(1, 5). \end{cases} \tag{36}$$

$$\textit{Case 2} \text{ – coupled current state: } \begin{cases} \overline{\delta} = \delta, \quad \overline{M}_{\overline{3}} = M_5, \\ \overline{p}_{t,t+1}(\overline{3}, \overline{1}) = p_{t,t+1}(5, 1) + p_{t,t+1}(5, 2), \\ \overline{p}_{t,t+1}(\overline{3}, \overline{2}) = p_{t,t+1}(5, 3) + p_{t,t+1}(5, 4), \\ \overline{p}_{t,t+1}(\overline{3}, \overline{3}) = p_{t,t+1}(5, 5). \end{cases} \tag{37}$$

We examine the consistency of the recovery results obtained by the two analysts by checking the holding of consistency condition (36) or (37) (Step 5).

Simulation results: For the single current state $\{\overline{1}\} = \{1\}$ (Case 1): Figure 2 plots the difference of the recovered time discount factors (top left panel) and the difference of the recovered transition probabilities (three remaining panels) by the two analysts. Each data point in these graphs corresponds to one simulation trial (of totally $N = 20$ simulation trials). The plotted differences are simulation counterparts of, and can be directly verified against, the consistency conditions (36).

The top left panel of Figure 2 provides simulation evidence that the recovered time discount factor δ and $\overline{\delta}$ are not equal. Despite of being quite small in the magnitude, a strictly non-zero difference $\overline{\delta} - \delta$ violates the consistency condition for the recovered time

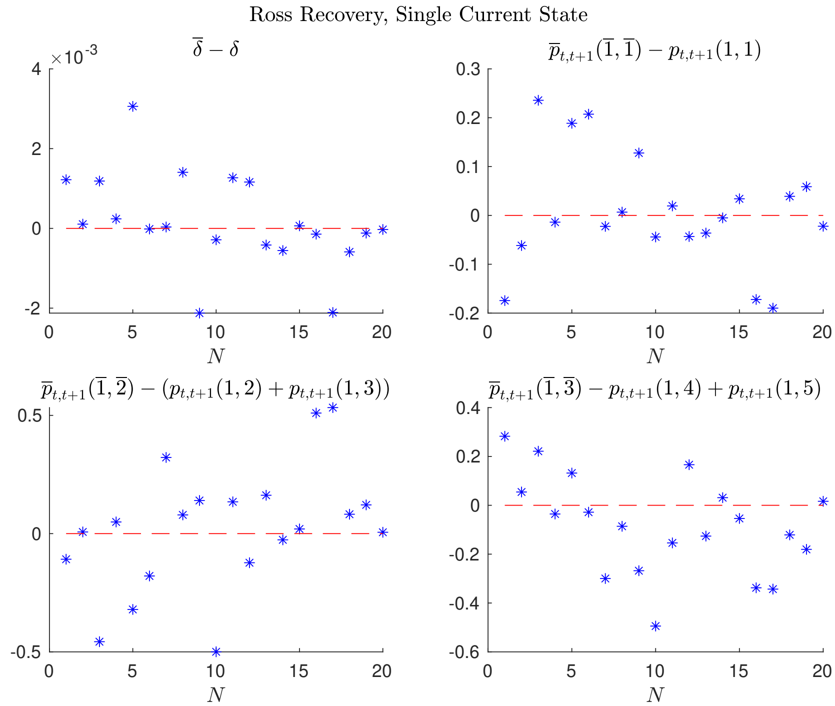


Figure 2: Case 1 – Single current state $\{\overline{1}\}=\{1\}$

discount factors in (36) because by design, the simulation environment is not subject to data measurement errors.[16] The remaining panels of Figure 2 also show broadly that the transition probabilities recovered by two analysts are inconsistent.

For the coupled current state $\{\overline{1}\}=\{1,2\}$ (Case 2): Similar to Case 1, the top left panel of Figure 3 provides simulation evidences that the recovered time discount factors δ and $\overline{\delta}$ are not equal. The remaining panels of Figure 3 also show broadly that the transition probabilities recovered by two analysts are inconsistent. In summary, our simulations show that Ross's recovery results under two different specifications are broadly inconsistent with each other whether the current state in the consolidated specification is a single or a coupled state.

[16]Instead, our simulation indicates that Ross's recovery procedure under the consolidated specification tends to produce time discount factors clustered around $\overline{\delta}\approx 1$ when the price data is generated from an objective (original) model with $\delta\approx 1$.

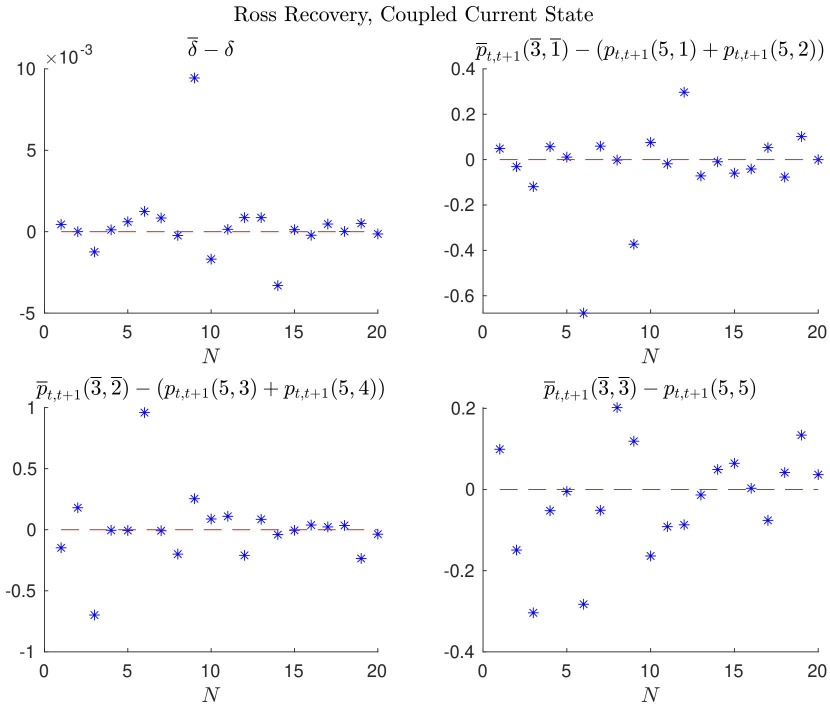


Figure 3: Case 2 – Coupled current state $\{\overline{1}\} = \{1, 2\}$

4 Generalized Recovery

This section presents a formal analysis and simulation results of various identification aspects of the generalized recovery. Our analysis again focuses on the comparison and consistency of the recovery results under different state space specifications. The comparative analysis setup is similar to that of the original Ross's recovery (Section 3) with an original specification $\mathcal{S} \equiv \{1, \ldots, S\}$ and a consolidated specification $\overline{\mathcal{S}} \equiv \{\overline{1}, \ldots, \overline{S}\}$. The consolidated specification is netted in the original one, $\overline{\mathcal{S}} \subset \mathcal{S}$, $\overline{S} < S$, and the associated mapping is described by an $S \times \overline{S}$ indicator matrix $\mathbf{C}$ (14). Our comparative analysis makes use of counting arguments concerning the nonlinear dynamics of the generalized recovery system (8), in contrast to the linear dynamics (4) and analysis of Ross's recovery. Section 4.1 analyzes JLP (2019)'s original generalized recovery approach, Section 4.2 presents a best-fit version of the generalized recovery, and Section 4.3 shows the simulation results.

4.1 Generalized Recovery: Exact Approach

Given an unobserved objective underlying market model to be recovered and a set of observed asset price data, two analysts implement the generalized recovery process respectively under their state space specifications. We consider separately two exhaustive scenarios, namely, the current original state is a single state or belongs to a coupled state in the consolidated specification.

Case 1: Single current state

In this case the current state is a single state. Specifically, let the first K consolidated states be single states and the remaining consolidated state be a coupled state (with $K = \overline{S} - 1$) as illustrated in Figure 4,

$$\overline{1} = \{1\}, \ldots, \overline{K} = \{K\}, \quad \overline{S} = \{K+1, \ldots, S\}.$$

Let the current state be the first state, so it is a single state $\overline{1} = \{1\}$.[17] Two analysts,

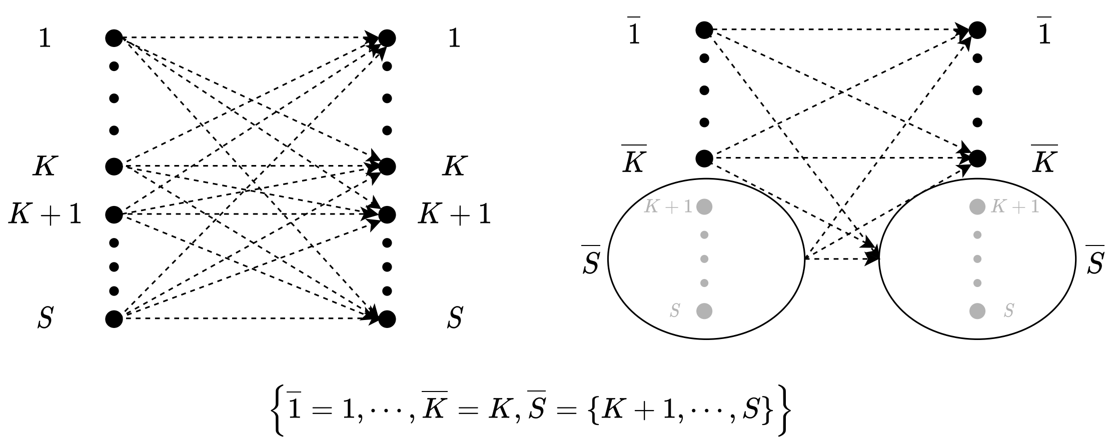


Figure 4: Case 1 – The current state is a single state: $\overline{1} = \{1\}$.

employing respectively the original and consolidated state space specifications, observe asset price data in the market but interpret AD assets contingent on their perceived specifications. As a result, AD asset prices observed and interpreted by the two analysts are related by the

[17] Example 1 illustrates this configuration when $\overline{1}$ is the current single state.

no-arbitrage principle as follows (similar to (17) for Ross's recovery),

$$\begin{cases} \text{Single states } \overline{j} = j \in \{1,\ldots,K\}: & \overline{A}_{\tau;\overline{1}\,\overline{j}} = A_{\tau;1j}, \\ \text{Coupled state } \overline{S}: & \overline{A}_{\tau;\overline{1}\,\overline{S}} = \sum_{j=K+1}^{S} A_{\tau;1j}, \end{cases} \quad \forall\tau \in \{1,\ldots,T\}, \tag{38}$$

where $A_{\tau;1j}$ and $\overline{A}_{\tau;\overline{1}\,\overline{j}}$ denote the prices of AD assets contingent respectively on states j and $\overline{j}$ of the original and consolidated specifications at time τ. While the first analyst solves the generalized recovery system (8) associated with the original specification, the second solves the following version of the same system but adapted to the consolidated specification,

$$\sum_{\overline{j}=\overline{1}}^{\overline{K}} \overline{A}_{\tau;\overline{1}\,\overline{j}} \frac{\overline{M}_{\overline{1}}}{\overline{M}_{\overline{j}}} + \overline{A}_{\tau;\overline{1}\,\overline{S}} \frac{\overline{M}_{\overline{1}}}{\overline{M}_{\overline{S}}} = \overline{\delta}^{\tau}, \quad \forall\tau \in \{1,\ldots,T\}. \tag{39}$$

Consistency conditions: These conditions are a set of relations that time and risk preferences and transition probabilities recovered for consolidated states need to satisfy to be consistent with their counterparts recovered for original states. The consistency conditions assure the compatibility and unambiguity of recovery results. We list and discuss these consistency conditions next and relegate their derivations to Internet Appendix A.2. The consistency conditions are,

$$\text{time discount factor}: \quad \overline{\delta} = \delta, \tag{40}$$

$$\text{transition probability}: \begin{cases} \overline{p}_{0,\tau}(\overline{1},\overline{j}) = p_{0,\tau}(1,j), & \forall\overline{j} = j \in \{1,\ldots,K\}, \\ \overline{p}_{0,\tau}(\overline{1},\overline{S}) = \sum_{\ell=K+1}^{S} p_{0,\tau}(1,\ell), & \forall\tau \in \{1,\ldots,T\}, \end{cases} \tag{41}$$

$$\text{marginal utilities}: \begin{cases} \frac{\overline{M}_{\overline{j}}}{\overline{M}_{\overline{1}}} = \frac{M_j}{M_1}, & \forall\overline{j} = j \in \{1,\ldots,K\}, \\ \frac{\overline{M}_{\overline{S}}}{\overline{M}_{\overline{1}}} = \sum_{\ell=K+1}^{S} \frac{p_{0,\tau}(1,\ell)}{\sum_{h=K+1}^{S} p_{0,\tau}(1,h)} \frac{M_\ell}{M_1}, & \forall\tau \in \{1,\ldots,T\}. \end{cases} \tag{42}$$

Discussion: Note that the marginal utility ratio $\frac{M_j}{M_1}$ is the state price density of state j under the original specification $\mathcal{S}$ (and $\frac{\overline{M}_{\overline{S}}}{\overline{M}_{\overline{1}}}$ the state price density of state $\overline{S}$ under the consolidated specification $\overline{\mathcal{S}}$).[18] The condition (42) requires that the recovered state price

[18]The pricing of AD asset implies that the ratio of marginal utilities equals the ratio of the AD price and the transition probability, $\delta\frac{M_j}{M_1} = \frac{A_{1j}}{p_{0,1}(1,j)}$, i.e., the price density.

density of the coupled state $\overline{S}$ be equal to the weighted average of the recovered state price density of the component original states, with the weights being the normalized probabilities, in order for the generalized recovery to be consistent under two different specifications. This consistency condition is intuitive. When ℓ is the more (less) likely state among all original states $\{K+1,\ldots,S\}$, i.e., the normalized probability $\frac{p_{0,\tau}(1,\ell)}{\sum_{h=K+1}^{S} p_{0,\tau}(1,h)}$ is large (small), the recovered state price density $\frac{M_\ell}{M_1}$ necessarily represents a more (less) important part in enforcing the consistency with the recovered state price density $\frac{\overline{M}_{\overline{S}}}{\overline{M}_{\overline{1}}}$ of the coupled state $\overline{S}$. We examine the tensions between consistency conditions and the exact generalized recovery approach next.

Tensions: We now examine the tensions between consistency conditions and the generalized recovery approach when the current state is a single state $\overline{1} = \{1\}$ illustrated in Figure 4.

Intuitively, on one hand, the consistency conditions (40)-(42) show that generalized recovery results under the original specification unambiguously imply unique generalized recovery results under the consolidated specification when these recovery results are consistent. This is due to the fact that the consolidated is netted in the original specification by construction (see (13)). On the other hand, recovery results under either specification are obtained from solving the nonlinear equation system adapted to the respective specification. These equation systems for different specifications are only loosely related via the asset prices observed by analysts. Therefore, the recovery solutions of the equation systems under the original and consolidated specifications in general do not satisfy the consistency conditions (40)-(42), indicating the inconsistency of the recovery results under the two specifications.

Quantitatively, we first assume that the generalized recovery works under the first analyst's (original) specification $\mathcal{S}$. Per JLP (2019), this means that the characteristics of the objective underlying market model are unambiguously recovered by obtaining an unique solution of the recovery equation systems (8).[19] The second analyst perceives the consolidated specification $\overline{\mathcal{S}}$, and then observes and employs AD asset prices $\overline{A}$'s (38) to solve the recovery system (39) associated with the consolidated specification. In the interest of a recovery consistency analysis, we assume that the generalized recovery also works under the second

[19]Recall that the key point of generalized recovery is that, when it works, the number of data horizons can be larger than the number of states, $T > S$, as discussed below (9).

analyst's (consolidated) specification $\overline{\mathcal{S}}$.[20] We then examine whether the associated recovery equation systems (8) and (39) are compatible with each other given that the conditions (40)-(42) hold as required for consistent recoveries. To this end, we derive a version of the consolidated recovery system that can be compared directly with the original recovery system. Specifically, we employ the AD price relation (38) and the required consistency of the time discount factors (40) and marginal utilities (42) to obtain the following version of the consolidated recovery system in terms of original AD asset prices A's,

$$\sum_{j=1}^{K} A_{\tau;1j} \frac{M_1}{M_j} + \sum_{j=K+1}^{S} A_{\tau;1j} \frac{\overline{M}_{\overline{1}}}{\overline{M}_{\overline{S}}} = \delta^{\tau}, \quad \forall \tau \in \{1, \dots, T\}. \tag{43}$$

We observe that the version (43) of the consolidated recovery system reduces to the original recovery system (8) when the latter has an identical marginal utility solution for every single state $\{j\}$ of the coupled state $\overline{S}$,[21]

$$\frac{M_1}{M_j} = \frac{\overline{M}_{\overline{1}}}{\overline{M}_{\overline{S}}}, \quad \forall j \in \overline{S} = \{K+1, \dots, S\}. \tag{44}$$

Recall that when the generalized recovery works for both specifications as we assumed, the original (8) and consolidated (43) recovery systems have *unique* solutions for *any* sufficiently large number of price data horizons $T > S$ (per JLP (2019), and the discussion below (9)). This uniqueness of the recovery solutions then implies that the equality of marginal utilities (44) in all single states $\{j\}$ corresponding to a coupled state $\overline{S}$ is both necessary and sufficient to reconcile the recovery results under the original and consolidated specifications. When this happens, condition (44) also simplifies and reconciles the consistent condition (42) for marginal utility of the consolidated state.[22] We recapitulate these findings in Proposition 2

[20]With regard to the consolidated specification, there are only two possibilities; either (i) the generalized recovery works (i.e., system (39) has a unique solution), or (ii) the generalized recovery does not work (i.e., system (39) has none or multiple solutions). We rule out the possibility (ii) in our analysis because in this case, the recovery works for the first but not the second analyst, making their recovery results outright incompatible.

[21]Note that factor $\frac{\overline{M}_{\overline{1}}}{\overline{M}_{\overline{S}}}$ is the same for all j's, so it was placed inside the second summation in the right-hand side of (43).

[22]After substituting the equality (44) of $\frac{M_j}{M_1}$ for all $j \in \overline{S} = \{K+1, \dots, S\}$ into (42), this consistency condition for the marginal utility of state $\overline{S}$ reduces to an identity, $\frac{\overline{M}_{\overline{S}}}{\overline{M}_{\overline{1}}} = \sum_{j=K+1}^{S} \frac{p_{0,\tau}(1,j)}{\sum_{h=K+1}^{S} p_{0,\tau}(1,h)} \frac{M_j}{M_1} =$

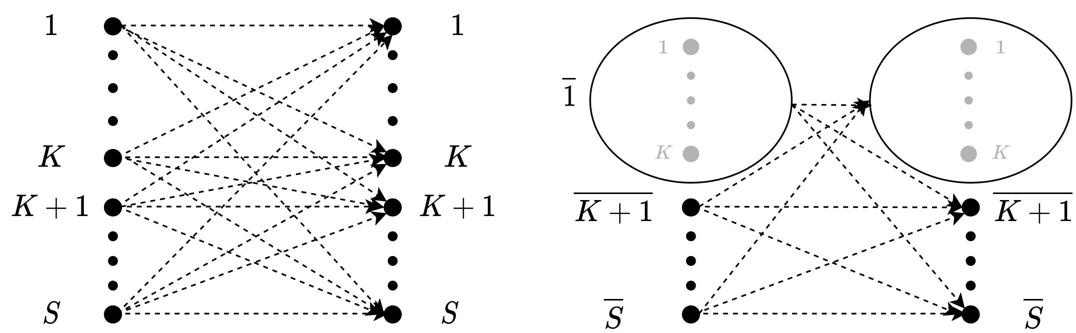
below, after discussing the alternative case of the coupled current state.

Case 2: Coupled current state

In this case, the current state is a coupled state. Specifically, let the first consolidated state $\overline{1}$ be a coupled state (composing of K first original states {1, ..., K}) and the remaining $\overline{S}-1$ consolidated states be single states (with $S-K=\overline{S}-1$) as illustrated in Figure 5,

$$\overline{1}=\{1,\ldots,K\},\ \ \overline{K+1}=\{K+1\},\ldots,\overline{S}=\{S\}.$$

Let the current state be the first state. It is the state $\{1\}$ to the first analyst who employs the original specification $\mathcal{S}$ and the state $\overline{1}$ to the second analyst who employs the consolidated specification $\overline{\mathcal{S}}$.[23] The observed AD asset prices are interpreted by the two analysts according



$$\left\{\overline{1}=\{1,\cdots,K\},\overline{K+1}=K+1,\cdots,\overline{S}=S\right\}$$

Figure 5: Case 2 – The current state is a coupled state: $\overline{1}=\{1,\ldots,K\}$.

to their perceived state space specifications and are related by the no-arbitrage principle as follows (similar to (38) for the case of single current state),

$$\begin{cases} \text{Coupled state } \overline{1}: & \overline{A}_{\tau;\overline{1}\,\overline{1}}=\sum_{j=1}^{K}A_{\tau;1j}, \\ \text{Single states } \overline{j}=j\in\{K+1,\ldots,S\}: & \overline{A}_{\tau;\overline{1}\,\overline{j}}=A_{\tau;1j}, \end{cases} \quad \forall\tau\in\{1,\ldots,T\}, \qquad (45)$$

$\frac{\overline{M}_{\overline{1}}}{\overline{M}_{\overline{S}}}\sum_{j=K+1}^{S}\frac{p_{0,\tau}(1,j)}{\sum_{h=K+1}^{S}p_{0,\tau}(1,h)}=\frac{\overline{M}_{\overline{1}}}{\overline{M}_{\overline{S}}}.$

[23] Example 1 illustrates this configuration when the coupled consolidated state $\overline{2}$ is the current state.

where $A_{\tau;1j}$ and $\overline{A}_{\tau;\overline{1}\,\overline{j}}$ denote the prices of AD assets contingent respectively on states j and $\overline{j}$ of the original and consolidated specifications at time τ. The first analyst solves the original generalized recovery system (8), and the second solves a similar system adapted to the consolidated specification,

$$\overline{A}_{\tau;\overline{1}\,\overline{1}} + \sum_{\overline{j}=\overline{K+1}}^{\overline{S}} \overline{A}_{\tau;\overline{1}\,\overline{j}} \, \frac{\overline{M}_{\overline{1}}}{\overline{M}_{\overline{j}}} = \overline{\delta}^{\tau}, \quad \forall \tau \in \{1, \ldots, T\}. \tag{46}$$

Consistency conditions: We present the consistency conditions and briefly explain their construction, before discussing their rationales and impacts on the recovery results under different specifications. The consistency conditions are as follows (similar to (40)-(42) for the single current state, and derived in Internet Appendix A.2),

$$\text{time discount factor :} \qquad \overline{\delta} = \delta, \tag{47}$$

$$\text{transition probability :} \begin{cases} \overline{p}_{0,\tau}(\overline{1},\overline{1}) = \sum_{\ell=1}^{K} p_{0,\tau}(1,\ell)\, \frac{M_\ell}{M_1}, & \forall \tau \in \{1,\ldots,T\}, \\ \overline{p}_{0,\tau}(\overline{1},\overline{j}) = p_{0,\tau}(1,j)\, \frac{1-\sum_{\ell=1}^{K} p_{0,\tau}(1,\ell)\, \frac{M_\ell}{M_1}}{1-\sum_{\ell=1}^{K} p_{0,\tau}(1,\ell)}, & \forall \overline{j} = j \in \{K+1,\ldots,S\}, \end{cases} \tag{48}$$

$$\text{marginal utilities :} \quad \frac{\overline{M}_{\overline{j}}}{\overline{M}_{\overline{1}}} = \frac{M_j}{M_1} \underbrace{\frac{1-\sum_{\ell=1}^{K} p_{0,\tau}(1,\ell)}{1-\sum_{\ell=1}^{K} p_{0,\tau}(1,\ell)\, \frac{M_\ell}{M_1}}}_{\equiv H}, \qquad \begin{array}{l} \forall \overline{j} = j \in \{K+1,\ldots,S\}, \\ \forall \tau \in \{1,\ldots,T\}. \end{array} \tag{49}$$

Discussion: The transition probability to the coupled state $\overline{1}$ under the consolidated specification (first equation in (48)) is an aggregation of the transition probabilities to the corresponding single states $j \in \overline{1} = \{1, \ldots, K\}$ under the original specification. Single states j of higher state prices $\left\{\frac{M_j}{M_1}\right\}$ are more prominent in this aggregation because they contribute more to the AD asset price $\overline{A}_{\tau;\overline{1}\,\overline{1}}$, based on which the second analyst infers the transition probabilities under the consolidated specification. By contrast, the consistency conditions for the transition probabilities to single states (second equation in (48)) require that these probabilities under the two specifications be proportional, with the proportionality being the same for all final single states $\overline{j} = j \in \{K+1, \ldots, S\}$. This is because both analysts observe identical AD contracts and prices that are contingent on these single states. The same reason justifies the proportionality of the state price densities of single states under the two specifications, leading to the consistency conditions (49) for marginal utilities.

Tensions: We now examine the tensions between consistency conditions and the generalized recovery approach when the current state is a couple state $\overline{1} = \{1, \ldots, K\}$ illustrated in Figure 5.

Intuitively, while consistency conditions (47)-(49) unambiguously imply unique consolidated recovery results from the original recovery results, the two set of results arise as solutions to two different recovery equation systems. These equation systems, being associated with original and consolidated specifications, are only loosely related via the asset prices observed by analysts. Therefore, the recovery results under the two specifications in general do not satisfy the consistency conditions (47)-(49), indicating the inconsistency of the recovery results under the two specifications.

Quantitatively, similar to the analysis of the tensions in the case of single current state (Case 1 above), we start with the assumption that the generalized recovery works for both specifications and the conditions (47)-(49) hold as required for the recovery consistency. In this premise, recovery results are unique solutions to the respective recovery equation systems (8) and (46). We then examine whether the systems (8) and (46) are compatible with each another given the required consistency conditions (47)-(49). To this end, we derive a version of the consolidated recovery system that can be compared directly with the original the recovery system. Specifically, we employ the AD price relation (45) and the required consistency of the time discount factors (47) and marginal utilities (49) to obtain the following version of the consolidated recovery system in terms of original AD asset prices A's,

$$\sum_{j=1}^{K} A_{\tau;1j} + \sum_{j=K+1}^{S} A_{\tau;1j} \frac{M_1}{M_j} H = \delta^{\tau}, \quad \forall \tau \in \{1, \ldots, T\}, \tag{50}$$

where $H = \frac{1-\sum_{j=1}^{K} p_{0,\tau}(1,j)}{1-\sum_{j=1}^{K} p_{0,\tau}(1,j) \frac{M_j}{M_1}}$ is defined in (49) and does not vary with state j. We observe that the version (50) of the consolidated recovery system reduces to the original recovery system (8) when the latter has an identical marginal utility solution for every single state

$\{j\}$ of the coupled state $\overline{1}$,[24]

$$\frac{M_1}{M_j} = 1, \quad \forall j \in \overline{1} = \{1, \dots, K\}, \tag{51}$$

As we assume that the generalized recovery works for both specifications, the original (8) and consolidated (50) recovery systems have *unique* solutions for *any* sufficiently large number of price data horizons $T > S$. This uniqueness of the recovery solutions then implies that the equality of marginal utilities (51) of all single states $\{j\}$ corresponding to a coupled state $\overline{1}$ is both necessary and sufficient to reconcile the recovery results under the original and consolidated specifications. This finding concerning the case of the coupled current state is similar to the one concerning the case of the single current state (44). We formalize these findings in the following result and relegate a detailed derivation to Internet Appendix B.2.

Proposition 2 *Let $\mathcal{S}$ and $\overline{\mathcal{S}}$ denote two state space specifications of the same objective but unobserved market model. The generalized recovery results obtained under the two specifications are consistent if and only if the marginal utilities of all single states $\{j\}$ under $\mathcal{S}$ that correspond to a coupled state $\overline{j}$ under $\overline{\mathcal{S}}$ are equal.*

$$\textit{Recoveries results under } \mathcal{S} \textit{ and } \overline{\mathcal{S}} \textit{ are consistent} \Longleftrightarrow M_i = M_k,\ \forall i, k \in \overline{j},\ \overline{j} \in \overline{\mathcal{S}}. \tag{52}$$

This necessary and sufficient condition holds regardless of whether the current state is a single or coupled state under the consolidated specification.

In the recovery process, asset prices are exogenous inputs; risk and time preferences and transition probabilities are endogenous outputs. While Proposition 2's condition (52) for consistent recoveries is stated in terms of the risk preferences, its restrictive nature is clear. The proposition asserts that, if the objective underlying market model does not feature equal marginal utilities in all states of $\mathcal{S}$ that belong to a consolidated state of $\overline{\mathcal{S}}$, then the generalized recoveries under two different specifications are inconsistent, even when these recoveries are unique under their respective specifications. An practical and key issue is that

[24] Note that when (51) holds, we have $H \equiv \frac{1-\sum_{j=1}^{K} p_{0,\tau}(1,j)}{1-\sum_{j=1}^{K} p_{0,\tau}(1,j)\ \frac{M_j}{M_1}} = 1$, and systems (50) and (8) are identical in this case. Therefore, we do not need the additional condition $H = 1$ to identify (50) with (8).

neither of the analysts observes the state space specification of the objective market model. Therefore, the choice of state specification is endogenous to the recovery process, so its associated generalized recovery results are subject to the inconsistency issue of Proposition 2. A comparison of conditions (27) and (52) indicates an identical (and strong) requirement for the consistency of Ross's and generalized recoveries (Propositions 1 and 2) under different state space specifications, namely, the equality of marginal utilities in all original states $j \in \mathcal{S}$ that correspond to a same consolidated state $\bar{j} \in \overline{\mathcal{S}}$.

4.2 Generalized Recovery: Best-fit Approach

The generalized recovery makes use of all T available (possibly redundant) horizons of price data, i.e., featuring more price constraints than unknowns. Such a system offers a unique solution, and the exact version of the general recovery works only when these AD price data arise consistently, hence redundantly, from the objective but unobserved market model (8) and the specification is correct (Proposition 2). To potentially mitigate these strong conditions, it is instructive to consider a best-fit version of the generalized recovery. Taking the recovered time discount factors as given, the best-fit approach simplifies the price constraints in the generalized recovery to regression equations. We examine whether the consistency issue underlying Proposition 2 can be addressed by this best-fit approximation of the generalized recovery.

Let $\{1\}$ and $\bar{1}$ be the current state under the original $\mathcal{S}$ and consolidated $\overline{\mathcal{S}}$ specifications. We stack the generalized recovery equations (8) for T different horizons under $\mathcal{S}$ and $\overline{\mathcal{S}}$ into respective matrix forms,

$$\underbrace{\begin{bmatrix} A_{11} & \dots & A_{1S} \\ A_{2;11} & \dots & A_{2;1S} \\ \vdots & \dots & \vdots \\ A_{T;11} & \dots & A_{T;1S} \end{bmatrix}}_{\mathbf{A}_\tau} \underbrace{\begin{bmatrix} 1 \\ \frac{M_1}{M_2} \\ \vdots \\ \frac{M_1}{M_S} \end{bmatrix}}_{\mathbf{f}} = \underbrace{\begin{bmatrix} \delta \\ \delta^2 \\ \vdots \\ \delta^T \end{bmatrix}}_{\boldsymbol{\delta}}, \quad \text{and} \quad \underbrace{\begin{bmatrix} \overline{A}_{\bar{1}\bar{1}} & \dots & \overline{A}_{\bar{1}\overline{S}} \\ \overline{A}_{2;\bar{1}\bar{1}} & \dots & \overline{A}_{2;\bar{1}\overline{S}} \\ \vdots & \dots & \vdots \\ \overline{A}_{T;\bar{1}\bar{1}} & \dots & \overline{A}_{T;\bar{1}\overline{S}} \end{bmatrix}}_{\overline{\mathbf{A}}_\tau} \underbrace{\begin{bmatrix} 1 \\ \frac{\overline{M}_{\bar{1}}}{\overline{M}_{\bar{2}}} \\ \vdots \\ \frac{\overline{M}_{\bar{1}}}{\overline{M}_{\overline{S}}} \end{bmatrix}}_{\bar{\mathbf{f}}} = \underbrace{\begin{bmatrix} \bar{\delta} \\ \bar{\delta}^2 \\ \vdots \\ \bar{\delta}^T \end{bmatrix}}_{\bar{\boldsymbol{\delta}}}, \tag{53}$$

where $\mathbf{A}_\tau$ denotes the $T \times S$ observed price matrix for T horizons (6) (with the current state $\{i\} \equiv \{1\}$), $\mathbf{f}$ the $S \times 1$ vector of marginal utility ratios $\left\{\frac{M_1}{M_j}\right\}$, $j \in \{1, \ldots, S\}$, and $\boldsymbol{\delta}$ the $T \times 1$ vector of powers of discount factors. The overline sign denotes the corresponding quantities under the consolidated specification. We assume that the price matrix $\mathbf{A}_\tau$ has full rank given the long time horizon and randomness in the price data.

Note that (i) the consistency condition for the recovered time discount factors requires that $\overline{\boldsymbol{\delta}} = \boldsymbol{\delta}$, and (ii) the nonlinearity in the generalized recovery equations (53) involves the powers of time discount factors. Therefore, at first we take the time discount factors and their required consistency, $\overline{\delta} = \delta$, as given. This step preserves the consistency requirement in the thought experiment with two subjective specifications while transforming (53) into linear systems of marginal utility ratios as unknowns, $\mathbf{A}_\tau \mathbf{f} = \boldsymbol{\delta}$ and $\overline{\mathbf{A}}_\tau \overline{\mathbf{f}} = \overline{\boldsymbol{\delta}}$. When price data of all horizons is employed and redundant, $T > S$ and $T > \overline{S}$, the best-fit recovery solutions of the marginal utilities are respectively,

$$\mathbf{f} = \left[\mathbf{A}_\tau' \mathbf{A}_\tau\right]^{-1} \mathbf{A}_\tau' \boldsymbol{\delta}, \qquad \overline{\mathbf{f}} = \left[\overline{\mathbf{A}}_\tau' \overline{\mathbf{A}}_\tau\right]^{-1} \overline{\mathbf{A}}_\tau' \overline{\boldsymbol{\delta}} = \left[\overline{\mathbf{A}}_\tau' \overline{\mathbf{A}}_\tau\right]^{-1} \overline{\mathbf{A}}_\tau' \boldsymbol{\delta}. \tag{54}$$

Using the relationship $\overline{\mathbf{A}}_\tau = \mathbf{A}_\tau \mathbf{C}$ (31) between AD matrices, we have[25]

$$\left[\mathbf{C}' \mathbf{A}_\tau' \mathbf{A}_\tau \mathbf{C}\right] \overline{\mathbf{f}} = \mathbf{C}' \mathbf{A}_\tau' \boldsymbol{\delta}. \tag{55}$$

We observe that the solution $\overline{\mathbf{f}}$ to the second equation in (54) (or equivalently (55)) does not present a valid marginal utility recovery result in general. This is because the first entry of such a solution generally differs from one, hence violating the normalization constraint of a valid recovery solution (per the second system in (53)). We state below a further result that identifies the existence of a valid and unique solution $\overline{\mathbf{f}}$ of (55) with the consistency of the generalized recovery's best-fit implementation and relegate a detailed derivation to Internet Appendix B.3.

Proposition 3 *Let $\mathcal{S}$ and $\overline{\mathcal{S}}$ denote two state space specifications of the same objective but*

[25] Indeed, (31) implies that the left-hand side of (55) is $\left[\mathbf{C}' \mathbf{A}_\tau' \mathbf{A}_\tau \mathbf{C}\right] \overline{\mathbf{f}} = \left[\overline{\mathbf{A}}_\tau' \overline{\mathbf{A}}_\tau\right] \overline{\mathbf{f}}$. The substitution of $\overline{\mathbf{f}}$ from (54) shows that this expression equals $\left[\overline{\mathbf{A}}_\tau' \overline{\mathbf{A}}_\tau\right] \left[\overline{\mathbf{A}}_\tau' \overline{\mathbf{A}}_\tau\right]^{-1} \overline{\mathbf{A}}_\tau' \boldsymbol{\delta} = \overline{\mathbf{A}}_\tau' \boldsymbol{\delta} = \mathbf{C}' \mathbf{A}_\tau' \boldsymbol{\delta}$, which is the right-hand side of (55).

unobserved market model. Suppose that the specification $\mathcal{S}$ is generated by AD asset prices, time preference, and marginal utilities denoted respectively as $\mathbf{A}_\tau$, $\boldsymbol{\delta}$, and $\mathbf{f}$. The best-fit generalized recoveries obtained under the two specifications $\mathcal{S}$ and $\overline{\mathcal{S}}$ are consistent if and only if the marginal utilities of all single states $\{j\}$ under $\mathcal{S}$ that correspond to a coupled state $\overline{j}$ under $\overline{\mathcal{S}}$ are equal.

$$\textit{Recoveries results under } \mathcal{S} \textit{ and } \overline{\mathcal{S}} \textit{ are consistent} \Longleftrightarrow M_i = M_k,\ \forall i, k \in \overline{j},\ \overline{j} \in \overline{\mathcal{S}}. \tag{56}$$

This necessary and sufficient condition holds regardless of whether the current state is a single or coupled state under the consolidated specification.

Recall that a successful recovery procedure requires a unique solution of the marginal utilities, probabilities, and time discount factor. In the best-fit recoveries above, the specification $\mathcal{S}$ is generated by the objective market model so that the first equation in (54) always holds by construction. Because we take the time preference consistency $\delta = \overline{\delta}$ as given to examine the consistency of the best-fit risk preference recovery, having a consistent and unique marginal utility solution $\overline{\mathbf{f}}$ to (55) is equivalent to obtaining a unique best-fit recovery for the specification $\overline{\mathcal{S}}$. To this end, Proposition 3 formally shows that (i) consistent recoveries under $\mathcal{S}$ and $\overline{\mathcal{S}}$ will always satisfy the best-fit equations (54), and (ii) if the best-fit solution $\overline{\mathbf{f}}$ is valid, then the recoveries are consistent. Note that the same necessary and sufficient condition underlies Propositions 2 and 3. Therefore, when this condition holds, the exact and best-fit approaches deliver the same generalized recovery results. Intuitively, this is because both approaches employ all available horizons of price data which are redundant but consistent when the condition of Proposition 2 and 3 holds. However, the necessary and sufficient condition of Propositions 2 and 3 is strong, signifying that the consistency issue remains the same even when we adopt an approximate (best-fit) implementation of the generalized recovery. The earlier observation that, in general, the solution to (55) does not present a valid recovery result reinforces this consistency issue of the best-fit approach to the general recovery. This finding mirrors a similar consistency issue of the best-fit implementation of Ross's recovery (Section 3.3).

4.3 Generalized Recovery: Simulation Results

This section presents simulation evidences to illustrate the generalized recovery process under different state space specifications (Proposition 2). Our simulation again closely follows the thought experiment setup of Section 2.2 to evaluate the generalized recovery results under different specifications. The simulation steps are as follows.

Simulation Procedure: Our simulation consists of $N = 20$ independent random trials. For each trial, data inputs are constructed as follows. We randomly generate (i) the time discount factor δ and marginal utility ratios $\frac{M_j}{M_1}$, $\forall j \in 2, \ldots, S$, from continuous uniform distributions respectively with supports $[0.8, 1]$ and $\left[\frac{1}{2}, 2\right]$, and (ii) the τ-period transition probabilities $\{p_{0,\tau}(1, j)\}$ from the current state $\{1\}$ to all future states j at time τ, where $\tau \in \{1, \ldots, T\}$. For the generalized recovery (8), we choose $T = S$, and make sure that for every $\tau \in \{1, \ldots, T\}$, the sum of transition probabilities is constrained to be one, $\sum_j p_{0,\tau}(1, j) = 1$. For each trial we perform the following five steps and report the corresponding simulation results.

Step 1: We take as given at first the characteristics of the underlying market model: (i) the state price specification $\mathcal{S} = \{1, \ldots, S\}$, and (ii) the time discount factor δ, the marginal utility ratios $\left\{\frac{M_j}{M_1}\right\}$ in every state $j \in \mathcal{S}$, and the τ-period transition probabilities $\{p_{0,\tau}(1, j)\}$ from the current state $\{1\}$ to all future states j at time τ, where $\tau \in \{1, \ldots, T\}$, $T = S$. All items in (ii) are generated in a random trial as described above.

Step 2: Using the above knowledge of objective market model and the τ-period AD pricing equation (9), we generate the current τ-period AD asset prices $\{A_{\tau;1j}\}$ that pay off in all future states $j \in \mathcal{S}$ at maturities $\tau \in \{1, \ldots, T\}$.[26] From now on, we disregard the knowledge of the time and risk preferences and transition probabilities $\left\{\delta, \frac{M_j}{M_1}, p_{0,\tau}(1, j)\right\}$ of the objective underlying market model and employ only the prices $\{A_{\tau,1j}\}$ generated in this step.

Step 3: Employing the original specification $\mathcal{S}$ and AD asset prices $\{A_{\tau,1j}\}$, for $j \in \mathcal{S}$ and $\tau \in \{1, \ldots, T\}$, the first analyst recovers the time discount factor, marginal utilities, and the τ-period transition probabilities associated with the specification $\mathcal{S}$ in the generalized

[26] Specifically, equation (9) gives the τ-period AD asset prices, $A_{\tau;1j} = \delta^\tau p_{0,\tau}(1, j) \frac{M_j}{M_1}$, in terms of the time discount factor δ, the marginal utilities $\left\{\frac{M_j}{M_1}\right\}$, and the τ-period transition probabilities $p_{0,\tau}(1, j)$.

recovery process. The generalized recovery process is implemented by first solving the system (8) to obtain the time discount factor, marginal utilities, and then (9) for the transition probabilities $\{p_{0,\tau}(1,j)\}$ that start from the current state $\{1\}$.

Step 4: Employing the consolidated specification $\overline{\mathcal{S}}$ and the consolidation process (either (38) or (45)), we construct the consolidated AD asset prices $\{\overline{A}_{\tau,\bar{1}\bar{j}}\}$, for $\bar{j} \in \overline{\mathcal{S}}$ and $\tau \in \{1, \ldots, T\}$ (now with $T = \overline{S}$). Observing these prices, the second analyst then recovers the time discount factor, marginal utilities, and the transition probabilities associated with the specification $\overline{\mathcal{S}}$ in the generalized recovery process (solving either (39) or (46)). That is, we repeat Step 3 while replacing the original AD asset prices $\{A_{\tau,1j}\}$ by the consolidated AD asset prices $\{\overline{A}_{\tau,\bar{1}\bar{j}}\}$, where $\bar{1}$ denotes the current state from the second analyst's perspective. For completeness, we consider two possible (and exhaustive) scenarios concerning the current state $\bar{1}$ of the consolidated specification.

> *Case 1*: Current state $\bar{1}$ is a single state, i.e., it is identical to the current original state, $\bar{1} = \{1\} \in \mathcal{S}$.
>
> *Case 2*: Current state $\bar{1}$ is a coupled state, i.e., it corresponds to several original states, $\bar{1} = \{1, \ldots, j\} \in \mathcal{S}$.

Step 5: We compare the recovery results obtained in Step 3 for the original specification $\mathcal{S}$ and Step 4 for the consolidated specification $\overline{\mathcal{S}}$ and check their consistency conditions (either (40)-(42) or (47)-(49)).

Simulation Model: Our specific simulation adopts a specification of $S = 5$ states, $\mathcal{S} = \{1, 2, 3, 4, 5\}$, for the underlying market model and randomly generates inputs as in Step 1, and AD asset prices $\{A_{\tau;1j}\}$ as in Step 2. The first analyst perceives the specification $\mathcal{S}$ and uses these AD prices $\{A_{\tau;1j}\}$ for $T = 5$ horizons, $\tau \in \{1, 2, 3, 4, 5\}$, to recover the time discount factor and marginal utilities (solving (8)) and transition probabilities that starts from current state $\bar{1}$ (solving (9)) (Step 3). For every simulation trial, these recovery results are identical to the time discount factor, marginal utilities, and transition probability inputs of Step 1, validating the generalized recovery approach.

The second analyst perceives the a 3-state specification $\overline{\mathcal{S}} = \{1, 2, 3\}$. We consider two scenarios listed in (35) concerning the current state $\overline{1}$ of the consolidated specification, namely, $\{\overline{1}\} = \{1\}$, $\overline{2} = \{2,3\}$, $\overline{3} = \{4,5\}$ (Case 1), and $\{\overline{1}\} = \{1,2\}$, $\overline{2} = \{3,4\}$, $\overline{3} = \{5\}$ (Case 2). For each case, (i) we construct the consolidated AD asset prices $\{\overline{A}_{\tau,\overline{1}\,\overline{j}}\}$, for $\overline{j} \in \overline{\mathcal{S}}$ and $\tau \in \{1,\dots,T\}$, and (ii) using these AD prices $\{\overline{A}_{\tau,\overline{1}\,\overline{j}}\}$ for $T = 3$ horizons, $\tau \in \{1,2,3\}$, the second analyst implements the generalized recovery to recover the time discount factor, marginal utilities, and transition probabilities that starts from current state $\overline{1}$ associated $\overline{\mathcal{S}}$ (Step 4). For each case, we examine the consistency of recoveries by the two analysts by checking whether their recovery results satisfy the consistency conditions (either (40)-(42) or (47)-(49), Step 5).

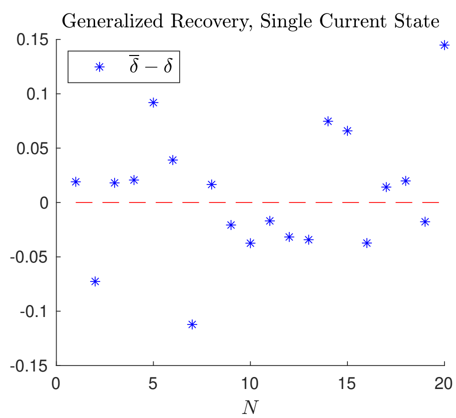


Figure 6: Single current state: Time discount comparison

Simulation results: For the single current state $\{\overline{1}\} = \{1\}$ (Case 1): Figure 6 plots the difference of the recovered time discount factors, and Figure 7 plots the root mean square error (RMSE) of the recovered transition probabilities (left panel) and of the recovered marginal utilities (right panel) by the two analysts.

The errors are defined as the absolute difference between simulation results of the left- and right-hand sides of each consistency equation in (40)-(42). For each simulation trial, the RMSE is computed using a panel of errors computed for different states and different maturities. Each data point in these graphs corresponds to one simulation trial (of totally $N = 20$ simulation trials). Figure 6 provides simulation evidence that the recovered time discount factors δ and $\overline{\delta}$ are not equal. Left and right panels of Figure 7 also show broadly that the transition probabilities and marginal utilities recovered by two analysts are inconsistent.

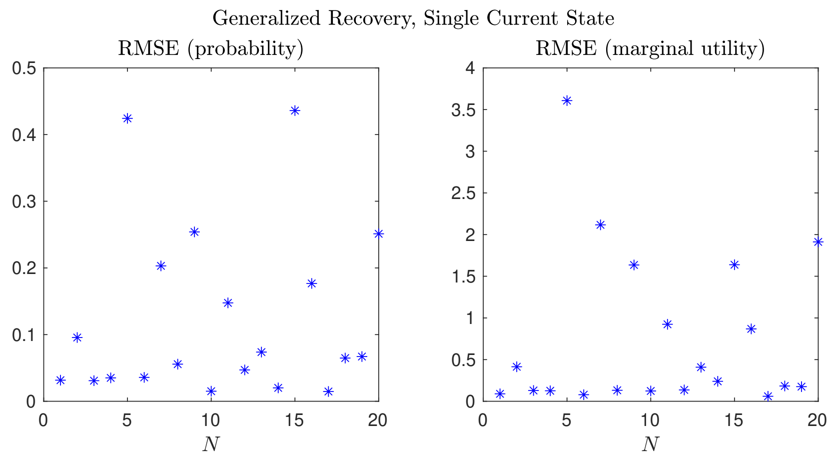


Figure 7: Single current state: RMSE of probabilities and marginal utilities

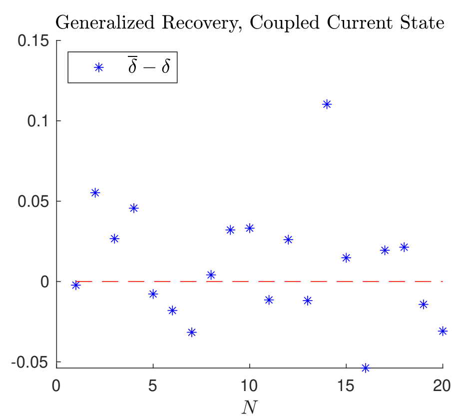


Figure 8: Coupled current state: Time discount comparison

For the coupled current state $\{\overline{1}\} = \{1, 2\}$ (Case 2): Similar to Case 1 (single current state), Figure 8 provides simulation evidence that the recovered time discount factors δ and $\overline{\delta}$ are not equal, and Figure 9 also shows broadly that the transition probabilities and marginal utilities recovered by two analysts are inconsistent. In summary, our simulations show that the generalized recovery results under two different specifications are broadly inconsistent with each other whether the current state in the consolidated specification is a single or a coupled state.

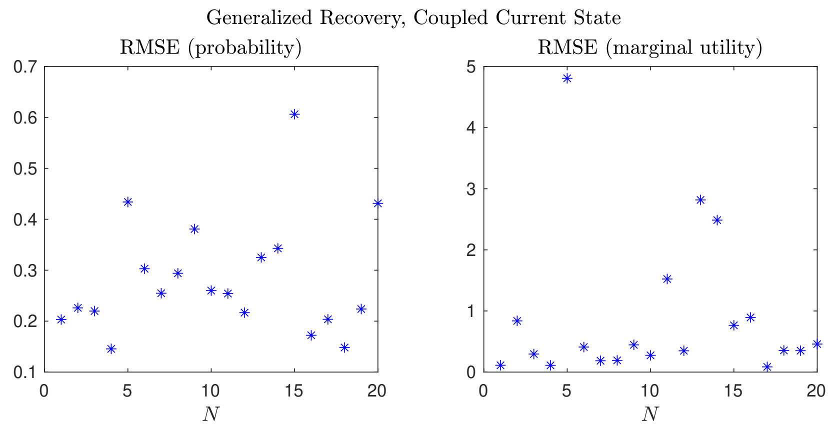


Figure 9: Coupled current state: RMSE of probabilities and marginal utilities

5 Continuous space-time setting

The analysis in previous sections shows that, for the same objective underlying market model, the recovery results associated with different discrete state space specifications are possibly irreconcilable even when these results are unique under the respective specification. We can have a deeper insight into the possible inconsistency of recovery results under different specifications from the perspective of a continuous state space setting, in which different discrete specifications emerge from different discretization schemes of the same underlying continuous state space specification. Given a set of price data of state-contingent financial assets observed in the financial markets, if the state space resolution resulted from the discretization does not match the state structure that available state-contingent assets can contract on, then the discretization scheme is incompatible with price data. Consequently, the recovery results based on such a discretization scheme may be inconsistent with those of the objective underlying market model. Our investigation then suggests a proper check for the state space discretization to alleviate this inconsistency issue. We derive the continuous-setting version of Ross's and generalized recovery equation systems in Section 5.1 and discuss their properties in Section 5.2.

5.1 Ross's and Generalized Recoveries in the Continuous Setting

We start out by assuming that the market is driven by a stochastic state variable Y_t in continuous time. For ease of exposition, we assume that Y_t is one-dimensional and follows some ex-ante unknown diffusion process in either physical or risk-neutral measure,[27]

$$\frac{dy_{t+dt}}{y_t} = \frac{y_{t+dt} - y_t}{y_t} = \mu_y dt + \sigma_y dB_t = \mu_y^Q dt + \sigma_y dB_t^Q, \tag{57}$$

where B_t and B_t^Q are standard Brownian motions in physical and risk-neutral measure respectively. The drifts μ_y, μ_y^Q and volatility σ_y are to be recovered ex-post and can be state-dependent (in form of processes adapted to natural filtration generated by B_t and B_t^Q).[28] All quantities in the market equilibrium are modeled as functions of this state variable, e.g., the SDF $M(y_t)$.

We recall that key to the recovery in the discrete state space setting is the AD asset price matrix $\mathbf{A}_t$, which is associated with a horizon length t. In this notation, the (i,j) element of $\mathbf{A}_{dt}$, i.e., $A_{dt,ij}$, is the current price of the AD security that pays \$1 only if the state in the time dt from now is j. In comparison, key to the recovery in the continuous state space setting is the infinitesimal operator $\mathcal{D}^Q$ associated with the risk-neutral state dynamic (57) because this operator plays the preeminent role of AD matrix in pricing all traded assets in the continuous setting, where $\mathcal{D}^Q$ is defined as

$$\mathcal{D}^Q \equiv y\mu_y^Q \frac{d}{dy} + \frac{1}{2} y^2 \sigma_y^2 \frac{d^2}{dy^2} - r(y), \tag{58}$$

and $r(y)$ is the short-term interest rate (or short rate) process. Formal correspondences between these two objects in both short and long time horizons are

$$\mathbf{A}_{dt} \longleftrightarrow \mathbb{1} + dt\mathcal{D}^Q; \qquad \mathbf{A}_t \longleftrightarrow e^{\int^t ds \mathcal{D}^Q}, \tag{59}$$

where the exponential operator $e^{\int^t ds \mathcal{D}^Q}$ should be understood as the power series of $\mathcal{D}^Q$.

[27] Higher dimensions and discontinuous dynamic (jumps) can also be incorporated.

[28] In special cases in which y is the price of traded assets such as equities or equity indexes, the risk-neutral drift coincides with risk-free rate, $\mu_y^Q = r(y)$.

These correspondences generalize an earlier result in Tran (2019). By resorting to the underlying continuous dynamics, the mapping (59) gives us a way to visualize and model the full AD price matrix for any time horizons, even for those that are shorter than what available in actual data. Furthermore, large and discontinuous changes in state variable y can also be conveniently modeled by incorporating jump dynamic into the infinitesimal operator $\mathcal{D}^Q$. An immediate application of the correspondence (59) is the construction of the recovery equations in the continuous state space setting. To this end, consider the eigenvalue problem of the AD price matrix, $\mathbf{A}\mathbf{x} = \delta\mathbf{x}$, where $\mathbf{x}$ denotes the eigenvector whose components $\{x_j\}$ are state-dependent in the discrete setting.

First, at the short-term horizon, this eigenvalue problem reduces to $\mathbf{A}_{dt}\mathbf{x} = \delta^{dt}\mathbf{x}$. In the limit of infinitesimal (continuous) time interval $dt \to 0$, an application of the correspondence (59) gives an continuous-time expression for this eigenvalue problem at the short-term horizon

$$\mathbf{A}_{dt}\mathbf{x} = \delta^{dt}\mathbf{x} \longleftrightarrow \left(\mathbb{1} + dt\mathcal{D}^Q\right)\mathbf{x}(y) = (\mathbb{1} + dt\log\delta)\,\mathbf{x}(y)$$

$$\text{or equivalently,} \;\; \mathcal{D}^Q\mathbf{x}(y) = -\rho\mathbf{x}(y), \quad \text{where: } \rho = -\log\delta, \;\; \delta = e^{-\rho}. \tag{60}$$

Note that ρ is the time discount rate in the continuous time. Using the expression (58) for the infinitesimal operator, the last equation is a differential equation in continuous setting

$$\frac{1}{2}y^2\sigma_y^2\frac{d^2\mathbf{x}(y)}{dy^2} + y\mu_y^Q\frac{d\mathbf{x}(y)}{dy} - r(y)\mathbf{x}(y) = -\rho\mathbf{x}(y). \tag{61}$$

Tran (2019) shows that all stochastic discount factors that are functions $M(y)$ of the state variable, of which Ross's recovery setting is a particular case, must satisfy the differential equation (61). Carr and Yu (2012) show that this differential equation (61) has *unique* positive solution $\mathbf{x}(y)$ (and the associated discount rate ρ) when (i) the state variable y is a bounded diffusion process, and (ii) appropriate Sturm-Liouville boundary conditions are imposed on the boundary of the support of y, thus signifying (61) as the recovery equation for Ross's recovery in the continuous setting.

Second, at long-term horizons, the eigenvalue problem $\mathbf{A}\mathbf{x} = \delta\mathbf{x}$ generalizes to

$$\mathbf{A}_t\mathbf{x} = \delta^t\mathbf{x}, \quad t \in \mathbb{R}^+. \tag{62}$$

When we fix the current state to be $\{1\}$, we limit the above matrix equation to component equations that concern only the first row of matrix $\mathbf{A}_t$, with varying horizons t. Observe that the elements of the matrix $\mathbf{A}_t$ are current prices of t-period AD assets. In particular, the first row of $\mathbf{A}_t$ contains the current AD asset prices $\{A_{t,1j}\}$ that initiate in the current state $\{1\}$ and payoff in states $\{j\}$ at t. Therefore, when being fixed to the current state $\{1\}$, (62) becomes the generalized recovery equation system (8). In the limit of infinitesimal (continuous) time interval $dt \to 0$, an application of the correspondence (59) gives a continuous-time expression for the generalized recovery equation (62)

$$\mathbf{A}_t\mathbf{x} = \delta^t\mathbf{x} \longleftrightarrow e^{\int^t ds\mathcal{D}^Q}\mathbf{x}(y) = e^{-\rho t}\mathbf{x}(y). \tag{63}$$

In summary, the same eigenvalue problem $\mathbf{A}\mathbf{x} = \delta\mathbf{x}$ of the AD price matrix reduces to (i) Ross's recovery in the short-term horizon when we do not limit the transitions to those starting only from the current state and (ii) the generalized recovery in long-term horizons when we limit the transitions to those starting only from the current state. We next discuss the implementation and consistency aspects of the recovery in the continuous setting.

5.2 Discussion

The recovery implementation of the recovery in the continuous setting starts with the discretization of the continuous state space, in which the difference between two discretization schemes corresponds to the difference between two state space specifications ($\mathcal{S}$ and $\overline{\mathcal{S}}$) of the discrete state space considered earlier. The discretization also yields an explicit mapping between the AD price matrix $\mathbf{A}$ and the infinitesimal operator (58).

The above analysis and the expression of the infinitesimal operator (58) show that the recovery processes in the continuous setting require the knowledge of the risk-neutral dynamics $\{\mu_y^Q, \sigma_y\}$ of the state variable and the short rate process $r(y)$. The risk-neutral dynamics replace the AD price matrix in the discrete-setting recovery processes (4) and (8) and can be inferred from option price data in model-independent approaches Breeden and Litzenberger (1978).[29]

For Ross's recovery process, these estimates of risk-neutral state dynamics $\{\mu_y^Q, \sigma_y\}$ from option prices are inputs into (61) to produce an explicit *linear* differential equation to recover uniquely the (inverse) marginal utility function $\mathbf{x}(y)$, after imposing appropriate boundary conditions. For example, in the special case in which the short rate is state-independent, $r(y) = r$, the unique positive eigenstate solution of (61) is a constant function $\mathbf{x}(y) = \mathbf{x}$, corresponding to the eigenvalue $\rho = r$. That is, the recovered time discount rate is risk-neutral when the short rate is state-independent. The same result is obtained by Ross (2015) in discrete state space setting. For the generalized recovery process, the same inputs of risk-neutral state dynamics into (63), however, do not produce a linear differential equation because generalized recovery dynamics are intrinsically nonlinear (8). To shed light into possible consistency issues of the recovery implementation discussed in previous sections (Proposition 1), we proceed with Ross's recovery in the continuous setting below.

To discretize the state space, we consider a finite difference representation of the infinitesimal operator,

$$\mathcal{D}^Q\mathbf{x}(y) = -r(y)\mathbf{x}(y) + \mu_y^Q y \frac{\mathbf{x}(y+dy) - \mathbf{x}(y-dy)}{2dy} + \frac{1}{2}\sigma_y^2 y^2 \frac{\mathbf{x}(y+dy) - 2\mathbf{x}(y) + \mathbf{x}(y-dy)}{(dy)^2}. \tag{64}$$

This representation, together with the correspondence $\mathbf{A}^{dt} \longleftrightarrow \left(\mathbb{1} + dt\mathcal{D}^Q\right)$, gives rise to

[29] More generally, when the state variable y (and the short rate r) is observable and markets are complete, a cross section of state-contingent contract prices identifies the risk-neutral state dynamic $\{\mu_y^Q, \sigma_y\}$. Bakshi et al. (2003) derive model-independent formulas for first four risk-neutral moments of state variables in term of integrals of option prices. Dubynskiy and Goldstein (2013) show that price $P(dY)$ of contract of payoff dy identifies μ_y^Q, and price $P(dy^2)$ of contract of payoff $(dy)^2$ identifies σ_y^2.

the following explicit matrix form for the AD matrix associated with horizon dt,

$$\mathbf{A}_{dt} = \begin{pmatrix} X & Y & 0 & 0 & \dots & 0 & 0 & 0 \\ Z & X & Y & 0 & \dots & 0 & 0 & 0 \\ 0 & Z & X & Y & \dots & 0 & 0 & 0 \\ \vdots & & & \ddots & & & & \vdots \\ \vdots & & & & \ddots & & & \vdots \\ 0 & 0 & 0 & \dots & Z & X & Y & 0 \\ 0 & 0 & 0 & \dots & 0 & Z & X & Y \\ 0 & 0 & 0 & \dots & 0 & 0 & Z & X \end{pmatrix}, \quad \text{where} \quad \begin{cases} X \equiv 1 - r(y)dt - \sigma_y^2 y^2 \frac{dt}{(dy)^2} \\ Y \equiv \frac{1}{2}\sigma_y^2 y^2 \frac{dt}{(dy)^2} + \frac{1}{2}\mu_y^Q y \frac{dt}{dy} \\ Z \equiv \frac{1}{2}\sigma_y^2 y^2 \frac{dt}{(dy)^2} - \frac{1}{2}\mu_y^Q y \frac{dt}{dy}. \end{cases} \tag{65}$$

Note that X, Y, and Z are functions of y, and thus their values vary with their positions in the AD price matrix.

To exemplify the possible causes of consistency issues found in discrete state setting, in what follows we assume that the market is driven by an underlying diffusion process y_t (57) with constant risk-neutral dynamics $\{\mu_y^Q, \sigma_y\}$ in the continuous time and state setting, though our analysis also applies to more general setting of state-dependent dynamics.

Arbitrage opportunities: The finite differencing representation (65) of AD price matrix $\mathbf{A}_{dt}$ with time horizon dt clearly shows that time and state discretization can give rise to negative AD prices and thus arbitrage opportunities. Indeed, X, Y, and Z in (65) are positive only when $\frac{dy}{y}$ and dt jointly satisfy

$$\left| \frac{\sigma_y^2}{\mu_y^Q} \right| > \frac{dy}{y} > \sigma_y \sqrt{\frac{dt}{1 - r(y)dt}}. \tag{66}$$

Analysts unaware of dynamic $\{\mu_y^Q, \sigma_y\}$ may unknowingly choose $\{dy, dt\}$ such that one (or both) of the above inequalities is violated. As a result, arbitrage opportunities may spuriously arise though they are absent in the original asset pricing model in the continuous setting. This analysis identifies a source of arbitrage opportunities that are artificially created in the process of discretizing time and state space. That is, the finite-difference step $\frac{dy}{y}$ of the state variable, i.e., the discretization of the state space, needs to be compatible with and bounded by the state $\{\mu_y^Q, \sigma_y\}$ and short rate $r(y)$ dynamics and the time step dt. In the state where the state variance is much smaller that the state growth rate, $\left|\frac{\sigma_y^2}{\mu_y^Q}\right| \ll 1$, one needs a fine discretization scheme (small $\frac{dy}{y}$). Intuitively, when the state dynamics are smooth, a high-resolution finite differencing is needed to recover the underlying state-contingent (inverse) marginal utility $\mathbf{x}(y)$. When the subjective discretization scheme by an analyst does not respect the bounds (66) imposed by the state dynamics, spurious arbitrage opportunities and inconsistent recovery results may arise.[30] Our analysis suggests that, in order to alleviate these spurious arbitrage opportunities, one needs to estimate the risk-neutral dynamics $\{\mu_y^Q, \sigma_y\}$ and short rate $r(y)$ and then choose a discretization scheme compatible with these estimates.

6 Conclusion

The implementation of recovery requires a subjective specification of the state space since this specification is not observed prior to the recovery process. We show that different subjective specifications may lead to permanent loss of information in price data, spurious arbitrages in the pricing of different effective AD matrices, and almost surely inconsistent recovered results of market's belief, time and risk preferences. From a perspective of the continuous time and state space setting, this recovery consistency issue is understood as arising from an improper and subjective discretization scheme of state space. In such a discretization scheme, the state space partition resulted from the discretization does not match the state structure that available state-contingent assets can contract on. As a result, the recovery results based on such a discretization scheme may be inconsistent with those of the objective underlying market model. We propose first to estimate the risk-neutral dynamics of state variables, which then informs a proper discretization of the time and state space in accordance with the sampling frequency of price data.

[30] To relate this finding with the possible inconsistency in the recovery results under two different discrete state space specifications (Propositions 1 and 2), the original specification $\mathcal{S}$ perceived by the first analyst corresponds to the objective underlying market model in the continuous setting. The consolidated specification $\overline{\mathcal{S}}$ perceived by the second analyst corresponds to a subjective discretization, which can lead to inconsistent recovery results if the subjective discretization scheme is incompatible with the state dynamics.

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Internet Appendix to "Recovery and Consistency"*

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A Consistency Conditions in Recovery

This section presents the derivation of the consistency conditions in Ross's and generalized recovery approaches. They are key to evaluate a recovery approach and are employed in the proofs of Propositions 1-3 in Section B below.

A.1 Ross's Recovery

An example

We consider the original specification $\mathcal{S} = \{1, 2, 3\}$. Suppose first that the current state $\{1\}$ is a single state, and the consolidated specification is $\overline{\mathcal{S}} = \{\overline{1}, \overline{2}\}$, where $\overline{1} = \{1\}$ and $\overline{2} = \{2, 3\}$. The no-arbitrage restrictions of AD assets are

$$\overline{A}_{\overline{1}\,\overline{1}} = A_{11}$$
$$\overline{A}_{\overline{1}\,\overline{2}} = A_{12} + A_{13},$$

which implies the following consistency conditions

$$\overline{p}_{t,t+1}(\overline{1}, \overline{1}) = p_{t,t+1}(1, 1)$$
$$\overline{p}_{t,t+1}(\overline{1}, \overline{2}) = p_{t,t+1}(1, 2) + p_{t,t+1}(1, 3)$$
$$\frac{\overline{M}_{\overline{2}}}{\overline{M}_{\overline{1}}} = \frac{\frac{M_2}{M_1} p_{t,t+1}(1, 2) + \frac{M_3}{M_1} p_{t,t+1}(1, 3)}{p_{t,t+1}(1, 2) + p_{t,t+1}(1, 3)}.$$

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Next, we consider the coupled current state. That is, the consolidated specification is $\overline{\mathcal{S}} = \{\overline{1}, \overline{2}\}$, where $\overline{1} = \{1, 2\}$ and $\overline{2} = \{3\}$. The no-arbitrage restrictions of AD assets are

$$\overline{A}_{\overline{1}\,\overline{1}} = A_{11} + A_{12}$$
$$\overline{A}_{\overline{1}\,\overline{2}} = A_{13},$$

which implies the following consistency conditions

$$\overline{p}_{t,t+1}(\overline{1}, \overline{1}) = p_{t,t+1}(1,1) + \frac{M_2}{M_1} p_{t,t+1}(1,2)$$
$$\overline{p}_{t,t+1}(\overline{1}, \overline{2}) = \frac{1}{H} p_{t,t+1}(1,3)$$
$$H = \frac{p_{t,t+1}(1,3)}{1 - p_{t,t+1}(1,1) - \frac{M_2}{M_1} p_{t,t+1}(1,2)}.$$

Case 1: Single current state

We consider the original specification of $\mathcal{S} = \{1, \cdots, S\}$ and the consolidated specification $\overline{\mathcal{S}} = \{\overline{1}, \overline{2}, \cdots, \overline{K}, \overline{S}\}$, where $\overline{1} = \{1\}, \overline{2} = \{2\}, \cdots, \overline{K} = \{K\}$, and $\overline{S} = \{K+1, \cdots, S\}$. Even though we only consider one coupled state here, it is straightforward to generalize the consistency conditions.

The no-arbitrage condition of an AD asset from the current state to another single state $\overline{j} = j \in \{1, \cdots, K\}$ is $\overline{A}_{\overline{1}\,\overline{j}} = A_{1j}$, which implies that $\overline{\delta} \frac{\overline{M}_{\overline{j}}}{\overline{M}_{\overline{1}}} \overline{p}_{t,t+1}(\overline{1}, \overline{j}) = \delta \frac{M_j}{M_1} p_{t,t+1}(1, j)$. Since the single states are identical in both the original and the consolidated specifications, we must have

$$\frac{\overline{M}_{\overline{j}}}{M_j} = \frac{\overline{M}_{\overline{1}}}{M_1}, \ \forall \overline{j} = j \in \{1, \cdots, K\}.$$

It is clear that the time discount factor $\overline{\delta}$ in the consolidated specification must be equal to the original time discount factor δ for the two specifications to be consistent. Together with the condition that $\overline{\delta} = \delta$, the above no-arbitrage restriction implies

$$\overline{p}_{t,t+1}(\overline{1}, \overline{j}) = p_{t,t+1}(1, j), \ \forall \overline{j} = j \in \{1, \cdots, K\} \text{ and } t.$$

Since $\sum_{\overline{j}\in\overline{\mathcal{S}}} \overline{p}_{t,t+1}(\overline{1},\overline{j}) = 1$, the probability $\overline{p}_{t,t+1}(\overline{1},\overline{S})$ must satisfy

$$\overline{p}_{t,t+1}(\overline{1},\overline{S}) = \sum_{j=K+1}^{S} p_{t,t+1}(1,j).$$

The no-arbitrage condition of an AD asset from the current state to the consolidated state $\overline{S}$ is $\overline{A}_{\overline{1}\,\overline{S}} = \sum_{j=K+1}^{S} A_{1j}$, i.e., $\overline{\delta}\frac{\overline{M}_{\overline{S}}}{\overline{M}_{\overline{1}}}\overline{p}_{t,t+1}(\overline{1},\overline{S}) = \sum_{j=K+1}^{S} \delta\frac{M_j}{M_1} p_{t,t+1}(1,j)$. Therefore,

$$\frac{\overline{M}_{\overline{S}}}{\overline{M}_{\overline{1}}} = \sum_{j=K+1}^{S} \frac{p_{t,t+1}(1,j)}{\sum_{i=K+1}^{S} p_{t,t+1}(1,i)} \frac{M_j}{M_1}.$$

Case 2: Coupled current state

We consider the original specification of $\mathcal{S} = \{1,\cdots,S\}$ and the consolidated specification $\overline{\mathcal{S}} = \{\overline{1},\overline{K+1},\cdots,\overline{S-1},\overline{S}\}$, where $\overline{1} = \{1,\cdots,K\}, \overline{K+1} = \{K+1\}, \overline{K+2} = \{K+2\},\cdots$, and $\overline{S} = \{S\}$. Even though we only consider one coupled state here, it is straightforward to generalize the consistency conditions.

The no-arbitrage condition of an AD asset from the current state to itself is $\overline{A}_{\overline{1}\,\overline{1}} = \sum_{j=1}^{K} A_{1j}$, which implies that $\overline{\delta}\frac{\overline{M}_{\overline{1}}}{\overline{M}_{\overline{1}}}\overline{p}_{t,t+1}(\overline{1},\overline{1}) = \sum_{j=1}^{K} \delta\frac{M_j}{M_1} p_{t,t+1}(1,j)$. Again, the time discount factor $\overline{\delta}$ in the consolidated specification must be equal to the original time discount factor δ for the two specifications to be consistent. Using the condition $\overline{\delta} = \delta$ gives

$$\overline{p}_{t,t+1}(\overline{1},\overline{1}) = \sum_{j=1}^{K} \frac{M_j}{M_1} p_{t,t+1}(1,j).$$

The no-arbitrage condition of an AD asset from the current state to another single state $\overline{j} = j \in \{K+1,\cdots,S\}$ is $\overline{A}_{\overline{1}\,\overline{j}} = A_{1j}$, which implies that $\overline{\delta}\frac{\overline{M}_{\overline{j}}}{\overline{M}_{\overline{1}}}\overline{p}_{t,t+1}(\overline{1},\overline{j}) = \delta\frac{M_j}{M_1} p_{t,t+1}(1,j)$. Thus, we have

$$\overline{p}_{t,t+1}(\overline{1},\overline{j}) = p_{t,t+1}(1,j)\frac{M_j}{M_1}\frac{\overline{M}_{\overline{1}}}{\overline{M}_{\overline{j}}}.$$

We know that $\frac{\overline{M}_{\overline{j}}}{M_j}$ must be the same for all $\overline{j} = j \in \{K+1,\cdots,S\}$ because the single states are identical under the original and the consolidated specifications. This means that $\frac{M_1}{\overline{M}_{\overline{1}}}\frac{\overline{M}_{\overline{j}}}{M_j}$ is the same across all single states. We denote this quantity as H.

Since $\overline{p}_{t,t+1}(\overline{1},\overline{1}) + \sum_{\overline{j}=\overline{K+1}}^{\overline{S}} \overline{p}_{t,t+1}(\overline{1},\overline{j}) = 1$, we have

$$\sum_{j=1}^{K} \frac{M_j}{M_1} p_{t,t+1}(1,j) + \sum_{j=K+1}^{S} p_{t,t+1}(1,j)\frac{1}{H} = 1,$$

which implies

$$H = \frac{1 - \sum_{j=1}^{K} p_{t,t+1}(1,j)}{1 - \sum_{j=1}^{K} \frac{M_j}{M_1} p_{t,t+1}(1,j)}$$

and

$$\overline{p}_{t,t+1}(\overline{1},\overline{j}) = p_{t,t+1}(1,j)\frac{1 - \sum_{\ell=1}^{K} p_{t,t+1}(1,\ell)}{1 - \sum_{\ell=1}^{K} \frac{M_\ell}{M_1} p_{t,t+1}(1,\ell)}, \quad \forall \overline{j} = j \in \{K+1,\cdots,S\}.$$

The above result also gives the consistency condition on marginal utilities of singles states

$$\frac{\overline{M}_{\overline{j}}}{\overline{M}_{\overline{1}}} = \frac{M_j}{M_1}\frac{1 - \sum_{\ell=1}^{K} p_{t,t+1}(1,\ell)}{1 - \sum_{\ell=1}^{K} \frac{M_\ell}{M_1} p_{t,t+1}(1,\ell)}, \quad \forall \overline{j} = j \in \{K+1,\cdots,S\}.$$

A.2 Generalized Recovery

Case 1: Single current state

We consider the original specification of $\mathcal{S} = \{1,\cdots,S\}$ and the consolidated specification $\overline{\mathcal{S}} = \{\overline{1},\overline{2},\cdots,\overline{K},\overline{S}\}$, where $\overline{1} = \{1\}, \overline{2} = \{2\},\cdots,\overline{K} = \{K\}$, and $\overline{S} = \{K+1,\cdots,S\}$. Even though we only consider one coupled state here, it is straightforward to generalize the consistency conditions. It is clear that the time discount factor $\overline{\delta}$ in the consolidated specification must be equal to the original time discount factor δ for the two specifications to be consistent.

For transition probabilities to another single state $\overline{j} = j \in \{1,\cdots,K\}$, we need to have $\overline{p}_{0,\tau}(\overline{1},\overline{j}) = p_{0,\tau}(1,j)$ because of the identical nature of single states. For the transition probability to the couple state $\overline{S}$, notice that

$$1 = \sum_{\overline{j} \in \overline{\mathcal{S}}} \overline{p}_{0,\tau}(\overline{1}, \overline{j}) = \sum_{\overline{j}=\overline{1}}^{\overline{K}} \overline{p}_{0,\tau}(\overline{1}, \overline{j}) + \overline{p}_{0,\tau}(\overline{1}, \overline{S}) = \sum_{j=1}^{K} p_{0,\tau}(1, j) + \overline{p}_{0,\tau}(\overline{1}, \overline{S}),$$

which implies that

$$\overline{p}_{0,\tau}(\overline{1}, \overline{S}) = 1 - \sum_{j=1}^{K} p_{0,\tau}(1, j) = \sum_{j=K+1}^{S} p_{0,\tau}(1, j).$$

For (ratios of) marginal utilities, we consider the no-arbitrage restrictions for AD assets. The prices of AD assets contingent on $\overline{j} = j \in \{1, \cdots, K\}$ must satisfy $\overline{A}_{\tau;\overline{1}\,\overline{j}} = A_{\tau;1j}$, which implies $\overline{\delta}^{\tau} \frac{\overline{M}_{\overline{j}}}{\overline{M}_{\overline{1}}} \overline{p}_{0,\tau}(\overline{1}, \overline{j}) = \delta^{\tau} \frac{M_j}{M_1} p_{0,\tau}(1, j)$. Using the consistency conditions for the time discount factor and transition probabilities, we have

$$\frac{\overline{M}_{\overline{j}}}{\overline{M}_{\overline{1}}} = \frac{M_j}{M_1}, \quad \forall \overline{j} = j \in \{1, \cdots, K\}.$$

The price of AD asset contingent on the coupled state $\overline{S}$ satisfies $\overline{A}_{\tau;\overline{1}\,\overline{S}} = \sum_{j=K+1}^{S} A_{\tau;1j}$, which implies that $\overline{\delta}^{\tau} \frac{\overline{M}_{\overline{S}}}{\overline{M}_{\overline{1}}} \overline{p}_{0,\tau}(\overline{1}, \overline{S}) = \sum_{j=K+1}^{S} \delta^{\tau} \frac{M_j}{M_1} p_{0,\tau}(1, j)$. Using the consistency conditions for the time discount factor and transition probabilities, we have

$$\frac{\overline{M}_{\overline{S}}}{\overline{M}_{\overline{1}}} = \sum_{j=K+1}^{S} \frac{p_{0,\tau}(1, j)}{\sum_{i=K+1}^{S} p_{0,\tau}(1, i)} \frac{M_j}{M_1}.$$

Case 2: Coupled current state

We consider the original specification of $\mathcal{S} = \{1, \cdots, S\}$ and the consolidated specification $\overline{\mathcal{S}} = \{\overline{1}, \overline{K+1}, \cdots, \overline{S-1}, \overline{S}\}$, where $\overline{1} = \{1, \cdots, K\}, \overline{K+1} = \{K+1\}, \overline{K+2} = \{K+2\}, \cdots$, and $\overline{S} = \{S\}$. Even though we only consider one coupled state here, it is straightforward to generalize the consistency conditions.

For the transition probability from the coupled current state $\overline{1}$ to itself, consider the no-arbitrage requirement that $\overline{A}_{\tau;\overline{1}\,\overline{1}} = \sum_{j=1}^{K} A_{\tau;1j}$, which implies that $\overline{\delta}^{\tau} \frac{\overline{M}_{\overline{1}}}{\overline{M}_{\overline{1}}} \overline{p}_{0,\tau}(\overline{1}, \overline{1}) = \sum_{j=1}^{K} \delta^{\tau} \frac{M_j}{M_1} p_{0,\tau}(1, j)$. Again, the time discount factor $\overline{\delta}$ in the consolidated specification must be equal to the original time discount factor δ for the two specifications to be consistent. Using the consistency condition $\overline{\delta} = \delta$ gives

$$\overline{p}_{0,\tau}(\overline{1},\overline{1}) = \sum_{j=1}^{K} \frac{M_j}{M_1} p_{0,\tau}(1,j).$$

For transition probabilities to single states $\overline{j} = j \in \{K+1, \cdots, S\}$, notice that $\overline{A}_{\tau;\overline{1}\overline{j}} = A_{\tau;1j}$, which implies

$$\overline{p}_{0,\tau}(\overline{1},\overline{j}) = p_{0,\tau}(1,j) \frac{M_j}{M_1} \frac{\overline{M}_{\overline{1}}}{\overline{M}_{\overline{j}}}.$$

We know that $\frac{\overline{M}_{\overline{j}}}{M_j}$ must be the same for all $\overline{j} = j \in \{K+1, \cdots, S\}$ because the single states are identical under the original and the consolidated specifications. This means that $\frac{M_1}{\overline{M}_{\overline{1}}} \frac{\overline{M}_{\overline{j}}}{M_j}$ is the same across all single states $\overline{j} = j \in \{K+1, \cdots, S\}$. We denote this quantity as H.

Since $\overline{p}_{0,\tau}(\overline{1},\overline{1}) + \sum_{\overline{j}=\overline{K+1}}^{\overline{S}} \overline{p}_{0,\tau}(\overline{1},\overline{j}) = 1$, using the earlier results we have

$$\sum_{j=1}^{K} \frac{M_j}{M_1} p_{0,\tau}(1,j) + \sum_{j=K+1}^{S} p_{0,\tau}(1,j) \frac{1}{H} = 1,$$

which implies

$$H = \frac{1 - \sum_{j=1}^{K} p_{0,\tau}(1,j)}{1 - \sum_{j=1}^{K} \frac{M_j}{M_1} p_{0,\tau}(1,j)}.$$

Thus, the transition probabilities to single states must satisfy

$$\overline{p}_{0,\tau}(\overline{1},\overline{j}) = p_{0,\tau}(1,j) \frac{1 - \sum_{\ell=1}^{K} p_{0,\tau}(1,\ell)}{1 - \sum_{\ell=1}^{K} \frac{M_\ell}{M_1} p_{0,\tau}(1,\ell)}, \quad \forall \overline{j} = j \in \{K+1, \cdots, S\}.$$

Finally, the above derivation also gives the consistency condition on marginal utilities

$$\frac{\overline{M}_{\overline{j}}}{\overline{M}_{\overline{1}}} = \frac{M_j}{M_1} \frac{1 - \sum_{\ell=1}^{K} p_{0,\tau}(1,\ell)}{1 - \sum_{\ell=1}^{K} \frac{M_\ell}{M_1} p_{0,\tau}(1,\ell)}, \quad \forall \overline{j} = j \in \{K+1, \cdots, S\}.$$

B Proofs

This section presents proofs of Propositions 1-3. These proofs center on verifying the holding of the consistency conditions derived in Section A.

B.1 Proof of Proposition 1

Case 1: Single Current State

We consider the original specification of $\mathcal{S} = \{1, \cdots, S\}$ and the consolidated specification $\overline{\mathcal{S}} = \{\overline{1}, \overline{2}, \cdots, \overline{K}, \overline{S}\}$, where $\overline{1} = \{1\}, \overline{2} = \{2\}, \cdots, \overline{K} = \{K\}$, and $\overline{S} = \{K+1, \cdots, S\}$. Suppose that the current state is $\overline{1}$ for the second (consolidated) analyst and $\{1\}$ for the first (original) analyst.

We first assume consistent recoveries under both specifications. The consistency requirements for this case based on the no-arbitrage restrictions of one-period AD assets are

$$\begin{aligned}
\overline{\delta} &= \delta \\
\frac{\overline{M}_{\overline{j}}}{M_j} &= \frac{\overline{M}_{\overline{1}}}{M_1}, \ \forall \overline{j} = j \in \{1, \cdots, K\} \\
\overline{p}_{t,t+1}(\overline{1}, \overline{j}) &= p_{t,t+1}(1, j), \ \forall \overline{j} = j \in \{1, \cdots, K\} \text{ and } t \\
\overline{p}_{t,t+1}(\overline{1}, \overline{S}) &= \sum_{j=K+1}^{S} p_{t,t+1}(1, j), \ \forall t \\
\frac{\overline{M}_{\overline{S}}}{\overline{M}_{\overline{1}}} &= \sum_{j=K+1}^{S} \frac{p_{t,t+1}(1, j)}{\sum_{i=K+1}^{S} p_{t,t+1}(1, i)} \frac{M_j}{M_1}, \ \forall t.
\end{aligned}$$

The last condition is the no-arbitrage restriction of one-period AD assets from the current state to the coupled state $\overline{S}$, or $\overline{A}_{\overline{1}\,\overline{S}} = \sum_{j=K+1}^{S} A_{1j}$, the pricing equation (7) of AD assets, and the consistency of recovered one-period transition probabilities listed above. Generalizing the no-arbitrage restriction to τ-period AD assets produces $\overline{A}_{\tau;\overline{1}\,\overline{S}} = \sum_{j=K+1}^{S} A_{\tau;1j}$, which implies that

$$\frac{\overline{M}_{\overline{S}}}{\overline{M}_{\overline{1}}} = \sum_{j=K+1}^{S} \frac{p_{t,t+\tau}(1,j)}{\sum_{i=K+1}^{S} p_{t,t+\tau}(1,i)} \frac{M_j}{M_1}, \tag{67}$$

where $p_{t,t+\tau}(1,j)$ is the probability from state 1 at t to state j at $t+\tau$. Note that one-period ahead probabilities are time-independent (i.e., $p_{t,t+1}(1,j) = p_{t+s,t+s+1}(1,j)$, $\forall s,j$) in the premise of Ross's recovery, although the implied (recovered) τ-period ahead probabilities $p_{t,t+\tau}(1,j)$ are no longer time-independent.

We observe that the consistency condition (67) implied from the no-arbitrage restriction must hold for all $\tau \in \{1,\cdots,T\}$. However, given a sufficiently large number of horizons, $T > S$, the system (67) has more equations than unknowns. The system has no solution, i.e., (67) does not hold, unless it is trivial: $\overline{M}_{\overline{S}} = M_j$ for all $j \in \{K+1,\cdots,S\}$. Therefore, when recoveries are consistent under $\mathcal{S}$ and $\overline{\mathcal{S}}$, marginal utilities are identical for all original states that belong to a consolidated state: $M_i = M_k$ $\forall i,k \in \overline{j}$, $\overline{j} \in \overline{\mathcal{S}}$.

Next, in the other direction of the proof, suppose that marginal utilities are identical for all original states that belong to a consolidated state, or $M_i = M_k$ for all $i,k \in \overline{j}$, $\overline{j} \in \overline{\mathcal{S}}$. The no-arbitrage restriction for the τ-period AD asset from current state to a single state $j \in \{1,\cdots,K\}$ is $\overline{A}_{\tau;\overline{1}\,\overline{j}} = A_{\tau;1j}$, which implies

$$\overline{\delta}^{\tau} \frac{\overline{M}_{\overline{j}}}{\overline{M}_{\overline{1}}} \overline{p}_{t,t+\tau}(\overline{1},\overline{j}) = \delta^{\tau} \frac{M_j}{M_1} p_{t,t+\tau}(1,j),\ \forall j \in \{1,\cdots,K\} \text{ and } \forall \tau \in \{1,\cdots,T\}, \tag{68}$$

where $\overline{p}_{t,t+\tau}(\overline{1},\overline{j})$ is the probability from state $\overline{1}$ at t to state $\overline{j}$ at $t+\tau$ and is time-dependent, i.e., $\overline{p}_{t,t+\tau}(\overline{1},\overline{j}) \neq \overline{p}_{t+s,t+s+\tau}(\overline{1},\overline{j})$ in general.

Similarly, the no-arbitrage restriction for the τ-period AD asset from current state to the coupled state $\overline{S}$ is $\overline{A}_{\tau;\overline{1}\,\overline{S}} = \sum_{j=K+1}^{S} A_{\tau;1j}$, which implies

$$\begin{aligned} \overline{\delta}^{\tau} \frac{\overline{M}_{\overline{S}}}{\overline{M}_{\overline{1}}} \overline{p}_{t,t+\tau}(\overline{1},\overline{S}) &= \sum_{j=K+1}^{S} \delta^{\tau} \frac{M_j}{M_1} p_{t,t+\tau}(1,j) \\ &= \delta^{\tau} \frac{M}{M_1} \sum_{j=K+1}^{S} p_{t,t+\tau}(1,j),\ \forall \tau \in \{1,\cdots,T\}, \end{aligned} \tag{69}$$

where $M \equiv M_j$ for all $j \in \overline{S}$. In order to have proper probabilities under $\overline{\mathcal{S}}$, we also require

$$\sum_{\overline{j} \in \overline{\mathcal{S}}} \overline{p}_{t,t+\tau}(\overline{1}, \overline{j}) = 1, \ \forall \tau \in \{1, \cdots, T\}. \tag{70}$$

Since we require a unique recovery under for the consolidated specification $\overline{\mathcal{S}}$, the solutions to the time discount factor, probabilities, and ratios of marginal utilities must satisfy equations (68)-(70) for all $\tau \in \{1, \cdots, T\}$. There are $T \times (K+2)$ equations from (68)-(70) and need to solve for $2K+2$ unknowns ($K+1$ probabilities $\overline{p}_{t,t+1}(\overline{1}, \overline{j})$, K ratios of marginal utilities $\frac{\overline{M}_{\overline{j}}}{\overline{M}_{\overline{1}}}$, and one time discount factor $\overline{\delta}$).[1] As we require the recovery to hold for a sufficiently large number T of horizons, the number of equations exceeds the number of unknowns. In general, the system does not have a solution, i.e., recovery results under different specification are not reconcilable, unless the system and solutions are trivial. This implies that the time discount factor, probabilities, and marginal utilities must satisfy the consistency conditions. Hence, the recoveries under $\mathcal{S}$ and $\overline{\mathcal{S}}$ are consistent.

Case 2: Coupled Current State

We consider the original specification of $\mathcal{S} = \{1, \cdots, S\}$ and the consolidated specification $\overline{\mathcal{S}} = \{\overline{1}, \overline{K+1}, \cdots, \overline{S-1}, \overline{S}\}$, where $\overline{1} = \{1, \cdots, K\}, \overline{K+1} = \{K+1\}, \overline{K+2} = \{K+2\}, \cdots$, and $\overline{S} = \{S\}$. Suppose that the current state is $\overline{1}$ for the second (consolidated) analyst and $\{1\}$ for the first (original) analyst.

We first assume consistent recoveries under both specifications. The consistency require-

[1]Recall that the one-period probabilities are time-independent. Although the τ-period probabilities depend on time, they can be constructed from the one-period probabilities. Thus, we just need to solve for $\overline{p}_{t,t+1}(\overline{1}, \overline{j})$. Also, $\frac{\overline{M}_{\overline{j}}}{\overline{M}_{\overline{1}}} = 1$ when $\overline{j} = \overline{1}$ so that we do not need to solve for it.

ments for this case based on the no-arbitrage restrictions of τ-period AD assets are

$$\begin{aligned}
\overline{\delta} &= \delta \\
\overline{p}_{t,t+\tau}(\overline{1},\overline{1}) &= \sum_{j=1}^{K}\frac{M_j}{M_1}p_{t,t+\tau}(1,j),\ \forall\tau\in\{1,\cdots,T\} \\
\overline{p}_{t,t+\tau}(\overline{1},\overline{j}) &= p_{t,t+\tau}(1,j)\frac{1-\sum_{\ell=1}^{K}p_{t,t+\tau}(1,\ell)}{1-\sum_{\ell=1}^{K}\frac{M_\ell}{M_1}p_{t,t+\tau}(1,\ell)},\quad \forall\overline{j}=j\in\{K+1,\cdots,S\}\text{ and }\tau\in\{1,\cdots,T\} \\
\frac{\overline{M}_{\overline{j}}}{\overline{M}_{\overline{1}}} &= \frac{M_j}{M_1}\frac{1-\sum_{\ell=1}^{K}p_{t,t+\tau}(1,\ell)}{1-\sum_{\ell=1}^{K}\frac{M_\ell}{M_1}p_{t,t+\tau}(1,\ell)},\quad \forall\overline{j}=j\in\{K+1,\cdots,S\}\text{ and }\tau\in\{1,\cdots,T\},
\end{aligned}$$

where $p_{t,t+\tau}(1,j)$ is the probability from state 1 at t to state j at $t+\tau$ and $\overline{p}_{t,t+\tau}(\overline{1},\overline{j})$ is the probability from state $\overline{1}$ at t to state $\overline{j}$ at $t+\tau$, both time-dependent.

In order to have proper probabilities under $\overline{\mathcal{S}}$, we need

$$\sum_{\overline{j}\in\overline{\mathcal{S}}}\overline{p}_{t,t+\tau}(\overline{1},\overline{j})=1,\ \forall\tau\in\{1,\cdots,T\}.$$

Using the above consistency conditions, we have

$$\sum_{j=1}^{K}\frac{M_j}{M_1}p_{t,t+\tau}(1,j)+\frac{1}{H}\sum_{j=K+1}^{S}p_{t,t+\tau}(1,j)=1,\ \forall\tau\in\{1,\cdots,T\}, \tag{71}$$

where $H=\frac{1-\sum_{j=1}^{K}p_{t,t+\tau}(1,j)}{1-\sum_{j=1}^{K}\frac{M_j}{M_1}p_{t,t+\tau}(1,j)}$ is required to be state- and time-independent.

From (71), we have T equations. In order to have these T equations to hold with time-invariant marginal utilities and also to keep H both state- and time-invariant, we must require $M_1=\cdots=M_K$ and, as a result, $H=1$. That is, we must have $M_i=M_k$ for all $i,k\in\overline{S}$. Therefore, when recoveries are consistent under $\mathcal{S}$ and $\overline{\mathcal{S}}$, marginal utilities are identical for all original states that belong to a consolidated state: $M_i=M_k\ \forall i,k\in\overline{j},\ \overline{j}\in\overline{\mathcal{S}}$.

Next, in the other direction of the proof, suppose that marginal utilities are identical for all original states that belong to a consolidated state, or $M_i=M_k$ for all $i,k\in\overline{j},\ \overline{j}\in\overline{\mathcal{S}}$. The no-arbitrage restriction for the τ-period AD asset from current state to a single state

$j \in \{K+1, \cdots, S\}$ is $\overline{A}_{\tau;\overline{1}\,\overline{j}} = A_{\tau;1j}$, which implies

$$\overline{\delta}^{\tau} \frac{\overline{M}_{\overline{j}}}{\overline{M}_{\overline{1}}} \overline{p}_{t,t+\tau}(\overline{1}, \overline{j}) = \delta^{\tau} \frac{M_j}{M_1} p_{t,t+\tau}(1, j), \ \forall j \in \{K+1, \cdots, S\} \text{ and } \forall \tau \in \{1, \cdots, T\}. \tag{72}$$

Similarly, the no-arbitrage restriction for the τ-period AD asset from the coupled current state $\overline{1}$ to itself is $\overline{A}_{\tau;\overline{1}\,\overline{1}} = \sum_{j=1}^{K} A_{\tau;1j}$, which implies

$$\overline{\delta}^{\tau} \overline{p}_{t,t+\tau}(\overline{1}, \overline{1}) = \sum_{j=1}^{K} \delta^{\tau} p_{t,t+\tau}(1, j), \ \forall \tau \in \{1, \cdots, T\}, \tag{73}$$

where we have used the assumption that $M_1 = \cdots = M_K$. In order to have proper probabilities under $\overline{\mathcal{S}}$, we need

$$\sum_{\overline{j} \in \overline{\mathcal{S}}} \overline{p}_{t,t+\tau}(\overline{1}, \overline{j}) = 1, \ \forall \tau \in \{1, \cdots, T\}. \tag{74}$$

Since we require a unique recovery under for the consolidated specification $\overline{\mathcal{S}}$, the solutions to the time discount factor, probabilities, and ratios of marginal utilities must satisfy equations (72)-(74) for all $\tau \in \{1, \cdots, T\}$. There are $T \times (S-K+2)$ equations from (72)-(74) and need to solve for $2(S-K+1)$ unknowns ($S-K+1$ probabilities $\overline{p}_{t,t+1}(\overline{1}, \overline{j})$, $S-K$ ratios of marginal utilities $\frac{\overline{M}_{\overline{j}}}{\overline{M}_{\overline{1}}}$, and one time discount factor $\overline{\delta}$). As we require the recovery to hold for a sufficient large number T of horizons, the number of equations exceeds the number of unknowns. In general, the system does not have a solution, i.e., recovery results under different specification are not reconcilable, unless the system and solutions are trivial. This implies that the time discount factor, probabilities, and marginal utilities must satisfy the consistency conditions. Hence, the recoveries under $\mathcal{S}$ and $\overline{\mathcal{S}}$ are consistent.

Finally, we note that there is only one coupled state in the consolidated specification in the above derivation. The proof concerning multiple coupled states proceeds with identical arguments and a straightforward modification of consolidation mapping between $\mathcal{S}$ and $\overline{\mathcal{S}}$.

B.2 Proof of Proposition 2

Case 1: Single Current State

We first assume that the recoveries are consistent under $\mathcal{S}$ and $\overline{\mathcal{S}}$ and we are given the recovered time discount factor, marginal utilities and probabilities under the original specification. As in Figure 4, we leave the first K states in the original specification are not consolidated. That is, the states in the consolidated model is $\overline{\mathcal{S}} = \{\overline{1}, \overline{2}, \cdots, \overline{K}, \overline{S}\}$, where $\overline{1} = \{1\}, \overline{2} = \{2\}, \cdots, \overline{K} = \{K\}$, and $\overline{S} = \{K+1, \cdots, S\}$. Since we require $T > S$, the consolidation $\overline{\mathcal{S}}$ would be trivial if $K+1$, i.e., $\mathcal{S} = \overline{\mathcal{S}}$ when $S = K+1$. Thus, we impose the number of states in the original specification $\mathcal{S}$ to be at least $K+2$, i.e., $S \geq K+2$. As a result, we have $T > S \geq K+2$.

Let $i \in \mathcal{S}$ and $\overline{i} \in \overline{\mathcal{S}}$. In the original specification, we define $h_i \equiv \frac{M_i}{M_1}$, and in the consolidated specification define $\overline{h}_{\overline{i}} \equiv \frac{\overline{M}_{\overline{i}}}{\overline{M}_{\overline{1}}}$. Note that by definition $h_1 = \overline{h}_{\overline{1}} = 1$.

Let $j = \overline{j} \in \{1, \cdots, K\}$ be the single states. According to the consistency condition (42),

$$\overline{h}_{\overline{j}} \equiv \frac{\overline{M}_{\overline{j}}}{\overline{M}_{\overline{1}}} = \frac{M_j}{M_1},$$

which implies

$$h_j = \overline{h}_{\overline{j}}, \ \forall j = \overline{j} \in \{1, \cdots, K\}. \tag{75}$$

Consider the coupled state $\overline{S}$. Specializing (7) to the consolidated specification yields

$$\overline{A}_{\tau;\overline{1}\,\overline{S}} = \overline{\delta}^{\tau} \overline{h}_{\overline{S}} \overline{p}_{0,\tau}(\overline{1}, \overline{S}) = \sum_{j=K+1}^{S} A_{\tau;1j} = \sum_{j=K+1}^{S} \delta^{\tau} h_j p_{0,\tau}(1, j), \ \forall \tau \in \{1, \cdots, T\}, \tag{76}$$

where the second equality is from (38) and the last one again follows from (7).

From the consistency conditions (40) and (41), we have $\overline{\delta} = \delta$ and

$$\overline{p}_{0,\tau}(\overline{1}, \overline{S}) = p_{0,\tau}(1, K+1) + \cdots + p_{0,\tau}(1, S), \quad \text{for all } \tau \in \{1, \cdots, T\}.$$

Thus,

$$\overline{A}_{\tau;\overline{1}\,\overline{S}} = \overline{\delta}^{\tau}\overline{h}_{\overline{S}}\overline{p}_{0,\tau}(\overline{1},\overline{S}) = \delta^{\tau}\overline{h}_{\overline{S}}\sum_{j=K+1}^{S} p_{0,\tau}(1,j),\ \forall\tau\in\{1,\cdots,T\}.$$

Comparing the above with (76), we have the following equation

$$\overline{h}_{\overline{S}} = \frac{\sum_{j=K+1}^{S} h_j p_{0,\tau}(1,j)}{\sum_{j=K+1}^{S} p_{0,\tau}(1,j)},\ \forall\tau\in\{1,\cdots,T\}. \tag{77}$$

Since we have T equations from (77) but only one unknown $\overline{h}_{\overline{S}}$, we will not have a solution in general unless equation (77) is trivial. That is, $h_{K+1} = h_{K+2} = \cdots = h_S = \overline{h}_{\overline{S}}$. Therefore, we must have $M_i = M_k$ for all $i,k\in\overline{S}$. Thus, we conclude that when recoveries are consistent under $\mathcal{S}$ and $\overline{\mathcal{S}}$, we have $M_i = M_k$ for all $i,k\in\overline{j},\ \overline{j}\in\overline{\mathcal{S}}$.

Next, in the other direction of the proof, suppose $M_i = M_k$ for all $i,k\in\overline{j},\ \overline{j}\in\overline{\mathcal{S}}$. Since we have set $M_1 = 1$, this is equivalent to letting $h\equiv h_{K+1} = h_{K+2} = \cdots = h_S$. We still assume that we are given the recovered time discount factor, marginal utilities, and probabilities under the original specification.

By no arbitrage, we must have $A_{\tau;1j} = \overline{A}_{\tau;\overline{1}\,\overline{j}}$ for all $j = \overline{j}\in\{1,\cdots,K\}$ and $\tau\in\{1,\cdots,T\}$. Equation (7) then implies that

$$\delta^{\tau}h_j p_{0,\tau}(1,j) = \overline{\delta}^{\tau}\overline{h}_{\overline{j}}\overline{p}_{0,\tau}(\overline{1},\overline{j}),\ \forall j=\overline{j}\in\{1,\cdots,K\} \text{ and } \forall\tau\in\{1,\cdots,T\}. \tag{78}$$

Similarly, for the couple state $\overline{S}$, we have $\overline{A}_{\tau;\overline{1}\,\overline{S}} = \sum_{j=K+1}^{S} A_{\tau;1j}$ and

$$\sum_{j=K+1}^{S} \delta^{\tau} h p_{0,\tau}(1,j) = \overline{\delta}^{\tau}\overline{h}_{\overline{S}}\overline{p}_{0,\tau}(\overline{1},\overline{S}),\ \forall\tau\in\{1,\cdots,T\}, \tag{79}$$

where we have used the assumption that $h = h_{K+1} = \cdots = h_S$. In addition, we require

$$\sum_{\overline{j}\in\overline{\mathcal{S}}} \overline{p}_{0,\tau}(\overline{1},\overline{j}) = 1, \forall\tau\in\{1,\cdots,T\}, \tag{80}$$

in order to have proper probabilities.

Since we require a unique recovery under for the consolidated specification $\overline{\mathcal{S}}$, the solutions to the time discount factor, probabilities, and ratios of marginal utilities must satisfy equations (78)-(80) for all $\tau \in \{1, \cdots, T\}$. There are $T \times (K+2)$ equations from (78)-(80) and need to solve for $(T+1)\times(K+1)$ unknowns ($T\times(K+1)$ probabilities $\overline{p}_{0,\tau}(\overline{1}, \overline{j})$, K ratios of marginal utilities $\overline{h}_{\overline{j}}$, and one time discount factor $\overline{\delta}$).As we have set $T > S \geq K+2$, the number of equations exceeds the number of unknowns. In general, the equations cannot be solved unless the solutions are trivial, implying that the time discount factor, probabilities, and marginal utilities satisfy (40)-(42). Hence, the recoveries under $\mathcal{S}$ and $\overline{\mathcal{S}}$ are consistent.

Case 2: Coupled Current State

We first assume that the recoveries are consistent under $\mathcal{S}$ and $\overline{\mathcal{S}}$ and we are given the recovered time discount factor, marginal utilities, and probabilities under the original specification. As in Figure 5, we consolidate the first K states and leave the rest $S-K$ states unchanged. That is, the states in the consolidated model is $\overline{\mathcal{S}} = \{\overline{1}, \overline{K+1}, \cdots, \overline{S-1}, \overline{S}\}$, where $\overline{1} = \{1, \cdots, K\}, \overline{K+1} = \{K+1\}, \overline{K+2} = \{K+2\}, \cdots$, and $\overline{S} = \{S\}$. In order to have a proper coupled state $\overline{1}$, we impose $K \geq 2$. As before, we also require $T > S$.

Let $i \in \mathcal{S}$ and $\overline{i} \in \overline{\mathcal{S}}$. In the original specification (7), we define $h_i \equiv \frac{M_i}{M_1}$. Similarly, in the consolidated model (46) we have $\overline{h}_{\overline{i}} \equiv \frac{\overline{M}_{\overline{i}}}{\overline{M}_{\overline{1}}}$, where again $\overline{1}$ denotes the consolidated state. By definition, $h_1 = \overline{h}_{\overline{1}} = 1$.

From the consistency condition (49), we know that for all states $\overline{j} = j \in \{K+1, \cdots, S\}$ that are not consolidated, the ratio of $\overline{h}_{\overline{j}}$ and h_j does not depend on the state, i.e.,

$$H \equiv \frac{\overline{h}_{\overline{j}}}{h_j} = \frac{\overline{M}_{\overline{j}}}{M_j}\frac{M_1}{\overline{M}_{\overline{1}}}, \ \forall j = \overline{j} \in \{K+1, \cdots, S\}. \tag{81}$$

Note that even if there is only one state that is not consolidated, equation (81) still holds. For probabilities, we express the second equation in (48) as

$$\overline{p}_{0,\tau}(\overline{1}, \overline{j}) = \frac{1}{H} p_{0,\tau}(1, j), \ \forall \tau \in \{1, \cdots, T\}. \tag{82}$$

We now move on to the states that are consolidated into state $\overline{1}$. Specializing equation

(7) to state $\overline{1}$ gives

$$\overline{A}_{\tau;\overline{1}\,\overline{1}}=\overline{\delta}^{\tau}\overline{p}_{0,\tau}(\overline{1},\overline{1})=\sum_{j=1}^{K}A_{\tau;1j}=\sum_{j=1}^{K}\delta^{\tau}h_{j}p_{0,\tau}(1,j), \tag{83}$$

where the second equality is from (45) and the last equation is from (7).

From the consistency conditions (47) and (48), we have $\overline{\delta}=\delta$ and $\overline{p}_{0,\tau}(\overline{1},\overline{1})=h_1p_{0,\tau}(1,1)+h_2p_{0,\tau}(1,2)+\cdots+h_Kp_{0,\tau}(1,K)$ for all $\tau\in\{1,\cdots,T\}$. Using the fact that $A_{\tau;1j}=\overline{A}_{\tau;\overline{1}\,\overline{j}}$ when $\overline{j}=j$ are not consolidated and (83), we can express (46) as

$$\delta^{\tau}\left(\sum_{j=1}^{K}h_jp_{0,\tau}(1,j)+\sum_{\overline{j}=j=K+1}^{S}\frac{h_j}{\overline{h}_{\overline{j}}}p_{0,\tau}(1,j)\right)=\overline{\delta}^{\tau},\ \forall\tau\in\{1,\cdots,T\},$$

which, together with the consistency condition (47), implies

$$\underbrace{\sum_{j=1}^{K}h_jp_{0,\tau}(1,j)}_{=\overline{p}_{0,\tau}(\overline{1},\overline{1})}+\underbrace{\frac{1}{H}\sum_{j=K+1}^{S}p_{0,\tau}(1,j)}_{=\sum_{\overline{j}=\overline{K+1}}^{\overline{S}}\overline{p}_{0,\tau}(\overline{1},\overline{j})}=1 \tag{84}$$

for all $\tau\in\{1,\cdots,T\}$. Since we have T equations from (84) but only one unknown H, we will not have a solution in general unless $h_1=h_2=\cdots=h_K=1$ and $H=1$. That is, we must have $M_i=M_k$ for all $i,k\in\overline{S}$. Thus, we conclude that when recoveries are consistent under $\mathcal{S}$ and $\overline{\mathcal{S}}$, we have $M_i=M_k$ for all $i,k\in\overline{j},\ \overline{j}\in\overline{\mathcal{S}}$.

Next, in the other direction of the proof, suppose $M_i=M_k$ for all $i,k\in\overline{j},\ \overline{j}\in\overline{\mathcal{S}}$. Since we have set $M_1=1$, this is equivalent to letting $h_1=h_2=\cdots=h_K=1$. We still assume that we are given the recovered time discount factor, marginal utilities, and probabilities under the original specification.

By no arbitrage, we must have $A_{\tau;1j}=\overline{A}_{\tau;\overline{1}\,\overline{j}}$ for all $j=\overline{j}\in\{K+1,\cdots,S\}$ and $\tau\in\{1,\cdots,T\}$. Equation (7) then implies that

$$\delta^{\tau}h_jp_{0,\tau}(1,j)=\overline{\delta}^{\tau}\overline{h}_{\overline{j}}\overline{p}_{0,\tau}(\overline{1},\overline{j}),\ \forall j=\overline{j}\in\{K+1,\cdots,S\}\text{ and }\forall\tau\in\{1,\cdots,T\}. \tag{85}$$

Similarly, for the couple state $\overline{1}$, we have $\overline{A}_{\tau;\overline{1}\,\overline{1}} = \sum_{j=1}^{K} A_{\tau;1j}$ and

$$\sum_{j=1}^{K} \delta^{\tau} p_{0,\tau}(1,j) = \overline{\delta}^{\tau} \overline{p}_{0,\tau}(\overline{1},\overline{1}),\ \forall \tau \in \{1,\cdots,T\}, \tag{86}$$

where we have used the assumption that $h_1 = \cdots = h_K = 1$ and $\overline{h}_{\overline{1}} = 1$. We also require

$$\sum_{\overline{j}\in\overline{\mathcal{S}}} \overline{p}_{0,\tau}(\overline{1},\overline{j}) = 1,\ \forall \tau \in \{1,\cdots,T\}, \tag{87}$$

in order to have proper probabilities.

Since we require a unique recovery under for the consolidated specification $\overline{\mathcal{S}}$, the solutions to the time discount factor, probabilities, and ratios of marginal utilities must satisfy equations (85)-(87) for all $\tau \in \{1,\cdots,T\}$. There are $T \times (S-K+2)$ equations from (85)-(87) and need to solve for $(T+1)\times(S-K+1)$ unknowns ($T\times(S-K+1)$ probabilities $\overline{p}_{0,\tau}(\overline{1},\overline{j})$, $S-K$ ratios of marginal utilities $\overline{h}_{\overline{j}}$, and one time discount factor $\overline{\delta}$). As we have set $K \geq 2$ and $T > S$, we have $T > S-K+1$, i.e., the number of equations exceeds the number of unknowns. In general, the equations cannot be solved unless the solutions are trivial, which implies that the time discount factor, probabilities, and marginal utilities must satisfy (47)-(49). Hence, the recoveries under $\mathcal{S}$ and $\overline{\mathcal{S}}$ are consistent.

Finally, we note that there is only one coupled state in the consolidated specification in the above derivation. The proof concerning multiple coupled states proceeds with identical arguments and a straightforward modification of consolidation map between $\mathcal{S}$ and $\overline{\mathcal{S}}$.

B.3 Proof of Proposition 3

Since $\overline{\boldsymbol{\delta}} = \boldsymbol{\delta}$ is a necessary condition for consistency, if $\mathcal{S}$ and $\overline{\mathcal{S}}$ are consistent, (55) becomes

$$\overline{\mathbf{f}} = \left[\mathbf{C}'\mathbf{A}'_{\tau}\mathbf{A}_{\tau}\mathbf{C}\right]^{-1}\mathbf{C}'\mathbf{A}'_{\tau}\boldsymbol{\delta} = \left[\mathbf{C}'\mathbf{A}'_{\tau}\mathbf{A}_{\tau}\mathbf{C}\right]^{-1}\mathbf{C}'\mathbf{A}'_{\tau}\mathbf{A}_{\tau}\mathbf{f},$$

which can be rearranged to

$$[\mathbf{C}'\mathbf{A}'_\tau\mathbf{A}_\tau\mathbf{C}]\,\overline{\mathbf{f}} = \mathbf{C}'\mathbf{A}'_\tau\mathbf{A}_\tau\mathbf{f}. \tag{88}$$

Recall that the ith entry of $\mathbf{f}$ is denoted as h_i^{-1} and the $\bar{i}$th entry of $\overline{\mathbf{f}}$ is $\overline{h_{\bar{i}}^{-1}}$.

Case 1: Single current state

We consider the original specification of $\mathcal{S} = \{1, \cdots, S\}$, which we are given, and the consolidated specification $\overline{\mathcal{S}} = \{\overline{1}, \overline{2}, \cdots, \overline{K}, \overline{S}\}$, where $\overline{1} = \{1\}, \overline{2} = \{2\}, \cdots, \overline{K} = \{K\}$, and $\overline{S} = \{K+1, \cdots, S\}$. Even though we only consider one coupled state here, it is straightforward to generalize the proof.

Suppose $\mathcal{S}$ and $\overline{\mathcal{S}}$ are consistent. Since the proof in Section B.2 does not restrict the time horizon T and the consistency conditions are derived from the no-arbitrage requirements, we know that $M_i = M_k,\ \forall i, k \in \bar{j},\ \bar{j} \in \overline{\mathcal{S}}$. The only difference is that under the OLS approach, the marginal utilities need to also satisfy (88).

We subtract the RHS from LHS in (88) and write out explicitly. The n-th row ($n \in \{1, \cdots, K\}$) is

$$R_n = \sum_{\tau=1}^{T} A_{\tau;1n}\left[\sum_{m=1}^{K} A_{\tau;1m}\overline{h_{\overline{m}}^{-1}} + (A_{\tau;1,K+1} + \cdots + A_{\tau;1S})\overline{h_{\overline{S}}^{-1}}\right] - \sum_{\tau=1}^{T} A_{\tau;1n}\sum_{m=1}^{S} A_{\tau;1m}h_m^{-1}$$

and the last row is

$$\begin{aligned} R_{K+1} = &\sum_{\tau=1}^{T}\left[(A_{\tau;1,K+1} + \cdots + A_{\tau;1S})\sum_{m=1}^{K} A_{\tau;1m}\overline{h_{\overline{m}}^{-1}} + (A_{\tau;1,K+1} + \cdots + A_{\tau;1S})^2\overline{h_{\overline{S}}^{-1}}\right] \\ &- \sum_{\tau=1}^{T}(A_{\tau;1,K+1} + \cdots + A_{\tau;1S})\sum_{m=1}^{S} A_{\tau;1m}h_m^{-1}. \end{aligned}$$

Because states $1, \cdots, K$ are single states, according to the consistency condition (42) the marginal utility ratios in those states must be the same under both specifications, i.e.,

$h_i = \overline{h}_{\overline{i}}$ for $i \in \{1, \cdots, K\}$. Therefore, the equations above for each row can be rewritten as

$$R_n = \sum_{\tau=1}^{T} A_{\tau;1n} \left[(A_{\tau;1,K+1} + \cdots + A_{\tau;1S}) \overline{h}_{\overline{S}}^{-1} - \sum_{m=K+1}^{S} A_{\tau;1m} h_m^{-1} \right], \; n \in \{1, \cdots, K\} \tag{89}$$

and

$$R_{K+1} = \sum_{\tau=1}^{T} (A_{\tau;1,K+1} + \cdots + A_{\tau;1S}) \left[(A_{\tau;1,K+1} + \cdots + A_{\tau;1S}) \overline{h}_{\overline{S}}^{-1} - \sum_{m=K+1}^{S} A_{\tau;1m} h_m^{-1} \right]. \tag{90}$$

Since $M_i = M_k$ for all $i, k \in \overline{j}, \overline{j} \in \overline{\mathcal{S}}$, we have $h_{K+1} = h_{K+2} = \cdots = h_S = \overline{h}_{\overline{S}}$. Hence, $R_1 = \cdots = R_{K+1} = 0$.

Next, suppose $M_i = M_k$ for all $i, k \in \overline{j}, \overline{j} \in \overline{\mathcal{S}}$. This is equivalent to letting $h_{K+1} = h_{K+2} = \cdots = h_S$. Again, the proof of Proposition 2 only relies on the no-arbitrage conditions, the result that the best-fit recoveries under $\mathcal{S}$ and $\overline{\mathcal{S}}$ are consistent still holds. It is then trivial to show that (88) holds.

Case 2: Coupled current state

We consider the original specification of $\mathcal{S} = \{1, \cdots, S\}$ and the consolidated specification $\overline{\mathcal{S}} = \{\overline{1}, \overline{K+1}, \cdots, \overline{S-1}, \overline{S}\}$, where $\overline{1} = \{1, \cdots, K\}, \overline{K+1} = \{K+1\}, \overline{K+2} = \{K+2\}, \cdots$, and $\overline{S} = \{S\}$. Even though we only consider one coupled state here, it is straightforward to generalize the proof.

Suppose $\mathcal{S}$ and $\overline{\mathcal{S}}$ are consistent. Since the proof in Section B.2 does not restrict the time horizon T and the consistency conditions are derived from the no-arbitrage requirements, we know that $M_i = M_k, \; \forall i, k \in \overline{j}, \; \overline{j} \in \overline{\mathcal{S}}$. The only difference is that under the OLS approach, the marginal utilities need to also satisfy (88).

We subtract the RHS from LHS in (88) and write out explicitly. The first row is

$$R_1 \equiv \sum_{\tau=1}^{T}\left[(A_{\tau;11}+\cdots+A_{\tau;1K})^2\overline{h}_{\overline{1}}^{-1}+(A_{\tau;11}+\cdots+A_{\tau;1K})\sum_{m=K+1}^{S}A_{\tau;1m}\overline{h}_{\overline{m}}^{-1}\right]$$
$$-\sum_{\tau=1}^{T}(A_{\tau;11}+\cdots+A_{\tau;1K})\sum_{m=1}^{S}A_{\tau;1m}h_m^{-1}.$$

The n-th row ($n \in \{K+1,\cdots,S\}$) is

$$R_n \equiv \sum_{\tau=1}^{T}A_{\tau;1n}\left[(A_{\tau;11}+\cdots+A_{\tau;1K})\overline{h}_{\overline{1}}^{-1}+\sum_{m=K+1}^{S}A_{\tau;1m}\overline{h}_{\overline{m}}^{-1}\right]-\sum_{\tau=1}^{T}A_{\tau;1n}\sum_{m=1}^{S}A_{\tau;1m}h_m^{-1}.$$

Because states $K+1,\cdots,S$ are single states, the marginal utilities in those states must be the same under both specifications, i.e., $h_i=\overline{h}_{\overline{i}}$ for $i \in \{K+1,\cdots,S\}$. Therefore, the equations above for each row can be rewritten as

$$R_1 = \sum_{\tau=1}^{T}(A_{\tau;11}+\cdots+A_{\tau;1K})\left[(A_{\tau;11}+\cdots+A_{\tau;1K})\overline{h}_{\overline{1}}^{-1}-\sum_{m=1}^{K}A_{\tau;1m}h_m^{-1}\right] \tag{91}$$

and

$$R_n = \sum_{\tau=1}^{T}A_{\tau;1n}\left[(A_{\tau;11}+\cdots+A_{\tau;1K})\overline{h}_{\overline{1}}^{-1}-\sum_{m=1}^{K}A_{\tau;1m}h_m^{-1}\right],\ n \in \{K+1,\cdots,S\}. \tag{92}$$

Also, since $h_i=h_k,\ \forall i,k \in \overline{j},\ \overline{j} \in \overline{\mathcal{S}}$, we know that $h_1=\cdots=h_K=\overline{h}_{\overline{1}}=1$, which implies that $R_1=0$. By similar arguments, $R_n=0$ for all $n \in \{K+1,\cdots,S\}$. Hence, the OLS equation (88) holds when $\mathcal{S}$ and $\overline{\mathcal{S}}$ are consistent.

Next, suppose $M_i=M_k$ for all $i,k \in \overline{j}, \overline{j} \in \overline{\mathcal{S}}$. This is equivalent to letting $h_1=h_2=\cdots=h_K$. Again, the proof of Proposition 2 only relies on the no-arbitrage conditions, the result that the best-fit recoveries under $\mathcal{S}$ and $\overline{\mathcal{S}}$ are consistent still holds. It is then trivial to show that (88) holds.